3d partition functions & 4d superconformal indices from 1d free fermions

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A thesis presented for the degree of
Doctor of Philosophy in Applied Mathematics

February 2016

Abstract

We study supersymmetric theories in three and four dimensions, focusing on two fundamental quantities - the partition function and the superconformal index. These observables give invaluable insight into the spectrum of supersymmetric theories and provide an important testing ground for dualities such as 3d mirror symmetry or AdS/CFT. As such, it is extremely desirable to evaluate them exactly. In recent years there has been immense progress in this direction: powerful exact techniques, such as supersymmetric localisation, have in many cases been able to reduce the calculation of these observables to the problem of evaluating finite dimensional matrix models. A great deal of effort has been applied to solving these matrix models, and already many results have been obtained. A particularly fruitful approach has been to manipulate these matrix models such that they resemble partition functions of free fermions in 1 dimension, so that one can apply powerful techniques from statistical mechanics.

We apply this 'Fermi gas approach' to study the matrix models of 3d $\hat{D}$ quiver theories with $U(N)$ gauge groups, as well as certain linear quiver theories, and we obtain large N evaluations of the partition function for a wide class of such theories. Along the way, we observe that 3d mirror symmetries - dualities that relate 3d supersymmetric theories that flow to the same infra-red fixed point - have a surprisingly elegant realisation in this setup, acting as linear canonical transformations on the Hamiltonians of the free fermions. This allows for extremely efficient derivations of the mirror maps - the relations between FI parameters and mass deformations on either side of the duality.

We also find a free fermion formulation for the matrix models corresponding to Schur indices of 4d $\hat{A}$ quiver theories with $SU(N)$ gauge groups, which allows us to extract their leading order behaviour at large N. For some special examples, including $\mathcal{N} = 4$ SYM, we are able to go further and obtain exact, all order evaluations of the Schur index.

Disclaimer: This thesis is based on work that has been published in [1–4]. The main body consists largely of an edited version of these four papers.

Acknowledgements

First and foremost, my thanks must go to my supervisor, Nadav Drukker. For his guidance and support throughout my time at King's I am extremely grateful. It is my pleasure to thank also Benjamin Assel and Jun Bourdier for fruitful collaborations and ever enjoyable and stimulating discussions. I am indebted to Nadav Drukker, Benjamin Assel and Asuka Kumon for reading this manuscript at various stages of completion and providing helpful criticism and comments.

My thanks too to all of my fellow PhD students, whose presence transformed an unremarkable office into a wonderful working environment. We have shared in each others' success and each others' frustrations: your support has been tremendous. This goes too, for my friends outside of academia, who have been a constant source of joy. I look forward to many years of shared enterprise with all of you.

Finally, I must thank my family, who have decorated the globe with places I can call home. You taught me to think freely, and your encouragement and support have helped me in every step of the way.

I am grateful for financial support received through an STFC studentship ST/K502066/1.

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Chapter 1

Introduction and statement of results

1.1 Introduction

Supersymmetric field theories have been the subject of intensive study for many years. Many beautiful results have been obtained, often going far beyond what could be achieved in the study of non supersymmetric theories. In this way, although supersymmetry is not evident in our own universe, such theories can still serve as important toy models: the techniques developed to study them will undoubtedly find applications in contexts more closely related to our own universe.

Supersymmetric theories are of course also interesting in their own right, and in studying them one often encounters wonderful and unexpected ties with other areas of physics and mathematics. One of the most celebrated of these relations is the conjectured AdS/CFT or gauge/gravity correspondence [5]. These remarkable dualities relate a gravitational quantum theory on AdS_n (n dimensional anti deSitter) space, with an $n-1$ dimensional CFT (conformal field theory) living on the boundary. Studying supersymmetric field theories can thus give us an insight into gravitational theories such as string or M-theory.

Here we study supersymmetric theories in three and four compact dimensions, concentrating on their partition functions and superconformal indices respectively. These fundamental quantities are among the first things one should like to compute when presented with a supersymmetric field theory - providing important insight into the spectrum of the theory, and acting as prerequisites to the evaluation of expectation values of observables such as Wilson loops. These quantities are also very natural to compare when testing dualities between field theories, or more exotic dualities such as AdS/CFT.

To compute partition functions of field theories generally requires the evaluation of an infinite dimensional path integral, a problem which in most cases can only be tackled perturbatively. A breakthrough came in [6], which showed that the path integral of $\mathcal{N}=4$ Super-Yang-Mills (SYM) theory on S^4 could be reduced by *exact* methods to a finite dimensional matrix model. Shortly thereafter this technique, known as supersymmetric localisation, was successfully applied also to the partition functions of $\mathcal{N}\geqslant 2$ Yang-Mills-Chern-Simons theories on S^3 [7–9]. The matrix models of S^3 partition functions that are obtained from localisation are in general still very difficult to solve, and initially progress was made through perturbative saddle point calculations that were able to probe the regime where the rank N of the gauge groups are taken to be large [10, 11]. This led eventually to a complete resummation of $\frac{1}{N}$ corrections in ABJM theory [12] in [13], which found that the perturbative part of the partition function takes the form of an Airy function

$$Z(N)=C^{-\frac{1}{3}}e^{A}\,\mathrm{Ai}\left[C^{-\frac{1}{3}}(N-B)\right]+Z_{\mathrm{np}}(N)\,, \tag{1.1}$$

where $Z_{\mathrm{np}}(N)$ denote nonperturbative, exponentially suppressed corrections. An important advance came with [14], who showed that the matrix models of ABJM and $\mathcal{N}\geqslant 3$ circular (or $\hat{A}$) quiver theories could in fact be reformulated as partition functions of N free fermions living on an infinite line. This invaluable observation facilitated the use of powerful techniques from statistical physics, which they used to analyse the 1d Fermi gas and elegantly reproduce the Airy function of ABJM (1.1). Furthermore they showed that this Airy function behaviour generalised to a large class of $\hat{A}$ quiver theories, with coefficients C, B and A depending on the details of the theory. This approach to solving the matrix models of 3d Chern-Simons theories became known as 'the Fermi gas approach', and has received intensive study yielding many exciting results, such as a complete understanding of nonperturbative corrections ($Z_{\mathrm{np}}(N)$ in(1.1)) in ABJ(M) in [15, 16], who built on the work of [17–28] (a full review can be found in [29]). The Fermi gas approach has also been successfully adapted to study a plethora of other 3d supersymmetric gauge theories, such as the aforementioned circular ($\hat{A}$) quivers in [1, 14, 30–34], $\hat{D}$ quivers in [2, 35], and theories with orthogonal/symplectic gauge groups in [2,36–38]. Supersymmetric Wilson loops have been studied in the Fermi gas approach in [39–41].

Our own work on the Fermi gas approach to partition functions of 3d theories, on which the first half of this thesis is based, has been published in [1] and [2].

In [1] we investigated 3d mirror symmetry from the viewpoint of the Fermi gas approach. Mirror symmetry, first proposed in [42], relates 3d theories with $\mathcal{N}=4$ supersymmetry that flow to the same IR fixed point, and whose moduli spaces are identical up to exchange of their Coulomb and Higgs branches [42–44]. This duality has a natural realisation in IIB string theory, where 3d $\mathcal{N}=4$ theories can be constructed as the worldvolume theory of stacks of D3 branes ending on NS5 or D5 branes [44, 45]. In this construction the R symmetry $SU(2)_L \times SU(2)_R$ is explicitly manifested as rotations of the internal directions of the NS5 and D5 branes which leave the system invariant. Mirror symmetry is manifested as a IIB S-duality transformation that converts D5 branes into NS5 branes and vice versa, exchanging the roles of $SU(2)_L$ and $SU(2)_R$.

Such brane constructions were also worked out for $\hat{D}$ quiver theories and theories with symplectic gauge groups by the inclusion of orientifold and orbifold planes [46–48], and this gave further evidence for a mirror duality between $\hat{D}$ quivers and symplectic linear quivers [43, 44].

In [49] it was proposed that mirror symmetry can be extended to an $SL(2, \mathbb{Z})$ web of dualities by including transformations that turn on Chern-Simons couplings for background gauge fields. These transformations do not commute with the conventional 'S' mirror duality, and thus give rise to additional mirror duals [50]. In the IIB construction this amounts to allowing for the full $SL(2, \mathbb{Z})$ group of generalised S-dualities, which transform D5 and NS5 branes into (p, q) five branes, giving rise to gauge theories with additional CS terms [51,52]. This was used to develop mirror pairs involving many more YM and CS theories, involving for example $\hat{A}$ quivers with gauge nodes of differing rank [53].

In [1] we considered the Fermi gas formulations of certain mirror dual $\mathcal{N} = 4$ $\hat{A}$ quiver theories, slightly generalising previous formulations by the inclusion of Fayet Iliopoulos (FI) and mass parameters. We found a surprisingly elegant relation: at the level of the Hamiltonians governing the fermions, mirror symmetry is realised by a simple canonical transformation exchanging position and momentum. Indeed, the $SL(2, Z)$ of mirror symmetry translated directly to an $SL(2, Z)$ group of linear canonical transformations. This allowed for an efficient derivation of the mirror maps - the relations between FI and mass parameters on either side of the duality.

In [2] we studied partition functions of $\mathcal{N} \geqslant 3$ $\hat{D}$ quiver theories and a family of linear quiver theories involving antisymmetric hypermultiplets and symplectic gauge groups. After developing some machinery to simplify the interactions between eigenvalues that occur in these matrix models, we again found a free fermion interpretation, this time with the fermions living on an infinite half line with boundary conditions at the origin.[1] Just as was observed for the $\hat{A}$ quivers we again found that mirror dual pairs of theories involving $\hat{D}$ quivers always gave rise to free fermions related by canonical transformation, and this again allowed an efficient derivation of the mirror maps. We were also able to extract the perturbative part of the partition function, which we found to take the form of (1.1). By developing an efficient

[1] A free fermion formulation of $\hat{D}$ quiver theories was developed concurrently by the authors of [35], whose results were consistent with ours.

recursive method we were able to evaluate C and B in (1.1) for a wide class of theories. Our results for the coefficient C agreed with previous results obtained in [54], which studied the matrix models of $\hat{D}$ quivers using saddle point techniques, while our results for B were completely new. These coefficients play an important role in allowing for nontrivial checks of the AdS/CFT correspondence, which predicts a duality between the 3d gauge theories we study and M-theory on $AdS_4 \times SE_7$. The SE_7 is a tri-Sasaki-Einstein manifold, whose precise structure depends on the details of the dual 3d gauge theory. It is well known [54] that the C coefficient should be directly related to the volume of the SE_7

$$C = \frac{6}{\pi^6}\,\mathrm{Vol}(SE_7)\,. \tag{1.2}$$

Indeed, our evaluation of C (which reproduces the result of [54]) bears out this prediction. However, the coefficient B is far less well understood in the context of AdS/CFT, and it is not known, even in the case of ABJM [55], how to evaluate it exactly on the gravity side. This remains as important future work.

The second half of this thesis is based on our papers [3, 4] and deals with a quantity closely related to the partition function, namely the superconformal index. This index is a generalisation of the Witten index [56] that was devised for 4d supersymmetric field theories in [57, 58]. The superconformal index counts states according to their quantum numbers and their fermionic/bosonic nature, and is given by the trace formula

$$\mathcal{I} = \mathrm{Tr}(-1)^F e^{-\beta\{Q,Q^\dagger\}} \prod_i \kappa_i^{G_i}\,. \tag{1.3}$$

Here Q and $Q^\dagger$ are a pair of conjugate supercharges and κ_i are fugacities for combinations of superconformal or flavour charges G_i that commute with both Q and $Q^\dagger$. In fact, by the usual Witten argument [56] it can be shown that the index receives net contributions only from states that are annihilated by $\{Q, Q^\dagger\}$, and is thus independent of β. It remains to count the states annihilated by $\{Q, Q^\dagger\}$ which contribute to the index depending on their quantum numbers under the charges G_i. This counting was carried out in [57,59], who thereby reduced the index to a matrix model.

The index coincides, up to an overall factor of the supersymmetric Casimir energy [60,61], with a partition function on $S^3 \times S^1$ (where the index fugacities are accounted for by imposing twisted boundary conditions on the S^1). Indeed, in [62–64] it was shown that, by taking the limit of the radius of S^1 going to zero, the index (after a suitable regularisation) precisely reproduces the S^3 partition function, with the index fugacities playing the role of squashing parameters or mass deformations. For the $\mathcal{N} \geqslant 2$ theories that we study, there is a special limit of the index known as the Schur index, which is an unrefined version of the superconformal index depending only on a single (non flavour) fugacity. The 3d reduction of this limit corresponds to turning off the squashing parameter, so that the 3d theory is on an unsquashed S^3. In this sense, the matrix models of 4d Schur indices are a natural generalisation of the matrix models of 3d partition functions studied in [14], for which the Fermi gas approach proved so powerful. This inspired us to try and solve the matrix models of the 4d Schur indices using Fermi gas techniques. Indeed, in [3] we were able to reformulate the Schur index of $U(N)$ $\mathcal{N} = 4$ SYM as a partition function of free fermions living on a circle. This gave us a great deal of control over the problem, and allowed us to evaluate the index exactly for arbitrary N as a power expansion in the fugacity. For finite N we were able to resum the expansion to obtain exact expressions involving complete elliptic integrals or Jacobi theta functions.

In [4] we considered the Schur indices of a family of theories that are a natural generalisation of $\mathcal{N} = 4$ SYM, namely $\mathcal{N} = 2$ circular ($\hat{A}$) quiver theories. We again found a free fermion interpretation, although this time with the added complication that the index was formulated as a weighted sum of an infinite number of free fermion partition functions. By analysing each gas separately and resumming the contributions at the end we were able to extract the leading order large N behaviour of the index for general $\hat{A}_{L-1}$ quivers (with gauge group $SU(N)^L$). For the $\hat{A}_1$ quiver we were able to push the analysis further, and obtain the index to all orders in N, as we did for $\mathcal{N} = 4$ SYM. We also obtained exact finite N evaluations of the index in terms of elliptic functions for quiver theories up to $\hat{A}_3$.

Again, an important utility of our results concerns AdS/CFT. The indices we compute should match (up to the aforementioned factor of the Casimir energy) with partition functions of string theory. However, it is a long standing puzzle to find a supergravity solution that gives a precise match, even for the leading order large N contribution. Our exact, all order evaluations of the index provide very precise predictions, that will hopefully be matched by supergravity calculations in the future.

1.2 Overview and statement of results

Here we summarise the content of the thesis, outlining the techniques we use and the results we obtain.

We begin chapter 2 by reviewing the 3d supersymmetric Chern Simons (CS) theories that we study, and the localisation technique that allows one to reduce the three sphere partition functions of $\mathcal{N} \geqslant 3$ quiver theories to matrix models. The end result is a simple set of rules that allow one to quickly associate a matrix model to a given quiver theory, which we summarise in section 2.3. In the Fermi gas approach, the integration variables λ_i of the matrix model are then interpreted as the positions of fermions on a line.

On the other hand, the partition function of a gas of N *free* fermions is given in terms of the corresponding one particle density operator $\hat{\rho}$

$$Z(N) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma \int d^N\lambda \prod_{i=1}^{N} \langle \lambda_i | \, \hat{\rho} \, | \lambda_{\sigma(i)} \rangle \, . \tag{1.4}$$

This structure is by no means self evident in the matrix models one obtains from supersymmetric localisation, and the first challenge in the Fermi gas approach is always to find a way to manipulate the matrix model into this form, a problem that we turn to in section 2.4.

For circular ($\hat{A}$) quivers the details were worked out in [14], and we provide a review of the procedure in section 2.4.1, generalising slightly by allowing for FI and mass deformations. The key ingredient is the application of a certain hyperbolic Cauchy determinant identity

$$\frac{\prod_{i<j} \sinh(\lambda_i - \lambda_j) \sinh(\tilde{\lambda}_i - \tilde{\lambda}_j)}{\prod_{i,j} \cosh(\lambda_i - \tilde{\lambda}_j)} = \sum_{\sigma \in S_N} (-1)^\sigma \prod_{i=1}^{N} \frac{1}{\cosh(\lambda_i - \tilde{\lambda}_{\sigma(i)})} \tag{1.5}$$

which factorises the matrix model integrand into a series of kernels that go around the quiver, and combine to give the density operator in (1.4).

For the matrix models of $\hat{D}$ quivers we worked out a Fermi gas formulation in [2], and the full details are presented in section 2.4.2. Again, the Cauchy determinant identity (1.5) allows the matrix model integrand to be factored into a number of kernels, but how they should combine into a structure like (1.4) is not immediately clear, due to the linear as opposed to circular quiver structure, and the branching at either end. The resolution involves coupling the kernels into a chain that traverses the quiver back and forth, and after taking care of some additional technical details we eventually find a formulation very close to (1.4). In fact, the expression we find can be interpreted as a partition function of free fermions living on a half line, with Neumann or Dirichlet boundary conditions at the origin.

An identical structure emerges [2] when we consider the matrix models of linear $U(N)$ quivers with additional symplectic gauge group factors or antisymmetric hypermultiplets, which we present in section 2.4.3. A special case of such quivers was previously studied in [36], and we discuss the relation between our and their Fermi gas formulations in appendix C.

For all of the classes of quiver theories we consider, the corresponding single particle density operators turn out to have remarkably elegant expressions in terms of canonical position and momentum operators $\hat{q}$ and $\hat{p}$. To give an example, the single particle density operator corresponding to single node $U(N)$ theory with a single adjoint and a single fundamental hypermultiplet is

$$\hat{\rho} = \frac{1}{\cosh \hat{p}} \frac{1}{\cosh \hat{q}} \,. \tag{1.6}$$

In section 2.5 we discuss 3d mirror symmetry from a Fermi gas perspective. As we observed in [1,2], mirror symmetry dualities are realised (for the examples we study) by simple canonical transformations acting in the density operator of the auxiliary fermions. Returning to the example of (1.6), applying a canonical transformation

$$\hat{p} \to \hat{p} + \hat{q} \,, \qquad \hat{q} \to -\hat{p} \tag{1.7}$$

gives

$$\hat{\rho} = \frac{1}{\cosh(\hat{p} + \hat{q})} \frac{1}{\cosh \hat{p}} \,, \tag{1.8}$$

which is nothing but the density operator associated with the mirror theory, ABJM. Indeed, applying such canonical transformations allows us to easily identify pairs of mirror theories and efficiently derive the corresponding mirror maps that relate mass parameters on one side of the duality with FI parameters on the other. We proceed to present a number of examples involving both $\hat{A}$ and $\hat{D}$ quivers, working out the mirror maps in each case.

In section 2.6 we return to the main objective of the Fermi gas approach, that is to solve matrix models. Having found a free Fermi gas formulation, a natural way to proceed [14,65] is to consider the grand canonical ensemble, and the associated grand potential $J(\mu)$, which has a simple expression in terms of spectral traces $Z_l = \operatorname{Tr} \hat{\rho}^l$

$$J(\mu) = \log\left[1 + \sum_{N=1}^{\infty} Z(N) e^{N\mu}\right] = -\sum_{l=1}^{\infty} \frac{(-1)^l Z_l}{l} \tag{1.9}$$

Following the approach of [14], we evaluate the spectral traces in Wigner phase space as an expansion in an auxiliary parameter ϵ, developing an efficient recursive method to treat very general $\hat{D}$ (and $\hat{A}$) quivers. This in turn leads to the large μ expansion of $J(\mu)$, which (as was found for $\hat{A}$ quivers already in [14]) takes the form

$$J(\mu) = \frac{C}{3}\mu^3 + B\mu + A + \mathcal{O}(e^{-\alpha\mu}) \,, \quad \alpha > 0 \,. \tag{1.10}$$

Substituting this into (1.9) and inverting immediately reproduces the Airy function behaviour (1.1) of the partition function. For $\hat{D}$ quivers without mass, FI or CS deformations but with arbitrary numbers of fundamental hypermultiplets we show (see appendix D) that the C and B coefficients receive contributions from only a finite number of orders in ϵ, and so we were able to evaluate them exactly

$$C = \frac{1}{4\pi^2 L\nu}, \qquad B = \frac{1 - 3\Delta\nu + 2\nu^2 + L(3+L)(\nu^2-1) - 3(\Sigma_1+\Sigma_2)}{12L\nu}. \tag{1.11}$$

The parameters $L, \nu, \Delta, \Sigma_1, \Sigma_2$ depend on the length of the quiver and on the number and distribution of fundamental hypermultiplets. Precise definitions can be found in section 2.6.3. The coefficient A receives contributions from all orders in ϵ, and we could not evaluate it exactly.

For completeness we also use our recursive approach to compute B and C also for $\hat{A}$ quivers in section 2.6.4, and our results agree with those obtained (for the mirror dual theories) in [33].

In chapter 3 we move on to the second major topic of this thesis, namely superconformal (Schur) indices of 4d theories. We begin in section 3.1 be reviewing the definition of 4d superconformal indices and the Schur limit. In particular the Schur index is given by the trace formula

$$\mathcal{I} = \mathrm{Tr}_{\mathcal{H}'}\,(-1)^F q^{2(E-R)} \prod_a e^{2iu^{(a)}F^{(a)}}\,, \tag{1.12}$$

where q and $e^{2iu^{(a)}}$ are fugacities for superconformal and flavour charges respectively.

In section 3.2 we then review the computation of [57,59], which were able to reduce the superconformal indices to matrix models involving special elliptic functions (for a list of definitions see appendix E).

In the remainder of chapter 3 we present the work of [3,4], in which we initiated the analysis of these matrix models using a Fermi gas approach, focusing on the Schur indices of $\hat{A}_{L-1}$ quiver theories (with gauge groups $SU(N)^L$ or $U(N)^L$).

As for the 3d partition functions, the first task is to manipulate the matrix models into the form of (1.4). To do this, we apply an elliptic generalisation of the Cauchy determinant identity (1.5) used in the 3d case. However, we are still left with some problematic interaction terms in the matrix model. We overcome this by Fourier expansion, generating an infinite sum of free Fermi partition functions whose density operators depend explicitly on the Fourier modes $\vec{n}$.

We proceed to study these Fermi gasses individually, and find that their associated grand partition functions admit a remarkably elegant expression as a product of Jacobi theta functions evaluated at the roots d_j of a certain polynomial

$$\Xi_{\vec{n}} = \frac{q^{|\vec{n}|^2}}{\vartheta_4^L \vartheta_3^L} \prod_{j=1}^{2L} \vartheta_3\big(d_j, q^{\frac{1}{2}}\big). \tag{1.13}$$

The order of this polynomial grows with the length L of the quiver and can only be factored algebraically for shorter quivers.

Nevertheless, as we present in section 3.4.1, we are able to make progress for arbitrary length $SU(N)$ $\hat{A}$ quivers by expanding the roots, and then the grand partition function asymptotically, from which we obtain the large N behaviour of the partition function, exact up to exponentially suppressed corrections. Finally by resumming the contributions from each Fermi gas, we obtain the leading order large N behaviour of the index

$$\mathcal{I}_{SU(N)} = \frac{q^{\frac{L}{6}}}{\eta^L(\tau)\eta^2(\frac{L\tau}{2})} + \mathcal{O}\big(e^{-cN}\big) . \tag{1.14}$$

In section 3.4.2 we reproduce this result in a different way, using an analytical continuation of the spectral traces Z_l to extract the large μ expansion of the grand potential (much like (1.10)).

As we mentioned before, the polynomial whose roots appear in (1.13) can be factored algebraically for shorter quivers, which gives completely explicit expressions for the grand potentials. As we present in section 3.5 this allows us to push the analysis further, and obtain exact large N expansions for the indices of the one node quiver (which is $\mathcal{N} = 4$ SYM) and the two node quiver

$$\begin{aligned}
\mathcal{I}^{L=1}_{SU(N)} &= \frac{q^{\frac{1}{4}}}{\eta^3(\tau)} \sum_{s=0}^{\infty} (-1)^s (N+2s) \frac{(N+s-1)!}{s!N!} q^{Ns+s^2} , \\
\mathcal{I}^{L=2}_{SU(N)} &= \frac{q^{\frac{1}{3}}}{\eta^4(\tau)} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} (N+k+l) \frac{(N+k-1)!(N+l-1)!}{N!(N-1)!k!l!} q^{N(k+l)+2kl} e^{2iuN(k-l)} .
\end{aligned} \tag{1.15}$$

Finally in section 3.6 we analyse the index of short quivers for finite values of N. Again the fact that we have completely explicit expressions for the grand potential of one and two node quivers is very powerful, and enables us to extract exact results in terms of complete elliptic integrals or Jacobi theta functions. For example, we find that the Schur index of a two node $SU(2)$ quiver is given by

$$\mathcal{I}^{L=2}_{SU(2)} = \frac{q^{-\frac{5}{3}}}{\eta^4(\tau)} \frac{-3E^2K^2 + 2(2-k^2)EK^3 - (1-k^2)K^4}{6\pi^4} , \tag{1.16}$$

where $K \equiv K(k^2)$ and $E \equiv E(k^2)$ are complete elliptic integrals of the first and second kind respectively and the elliptic modulus is $k = \vartheta_2^2/\vartheta_3^2$. By examining the finite N results we obtain for one and two node quivers, we conjecture an ansatz for the indices of longer quivers. On the other hand, for fixed N and L we can obtain the q expansion of the index to finite order. In section 3.6.4 we exploit this to obtain exact results also for longer quivers of up to four nodes.

Chapter 2

Sphere partition functions of 3d theories

In this chapter we study partition functions of $\mathcal{N} \geqslant 3$ Yang Mills (YM) and Chern Simons (CS) gauge theories on the three sphere. These partition functions were reduced to matrix models using supersymmetric localisation in [7–9] (see also [66] for a review). Our work focuses on the analysis of the matrix models, which we will come to in section 2.4

First, we will review the 3d theories we consider, before reviewing the localisation procedure in full detail. In section 2.3 we collect the results of localisation, and give the necessary group theory data that quickly allow one to associate a matrix model with a quiver gauge theory.

2.1 3d supersymmetric gauge theories

We formulate the theories we study in terms of $\mathcal{N} = 2$ multiplets that form representations of the super Poincaré algebra. These are the vector and chiral multiplets, whose component fields and supersymmetry variations we summarise below.

First a quick summary of conventions. Barred and unbarred spinors transform in the fundamental representation of the Lorentz group, and their indices are raised and lowered by the antisymmetric tensor

$$\bar{\epsilon}\lambda \equiv \bar{\epsilon}^{\alpha}\lambda_{\alpha} = \bar{\epsilon}^{\alpha}\varepsilon_{\alpha\beta}\lambda^{\beta} \tag{2.1}$$

γ_μ are the Pauli matrices, and $\gamma_{\mu\nu}$ denotes the antisymmetrisation

$$\gamma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] \tag{2.2}$$

2.1.1 Vector multiplets

An $\mathcal{N}=2$ vector multiplet has a gauge field A_μ, scalars σ and D, and fermions λ, $\bar{\lambda}$, which all transform in the adjoint representation of the gauge group.

Supersymmetry variations on S^3 are parameterised by two spinors ϵ, $\bar{\epsilon}$

$$
\begin{aligned}
\delta A_\mu &= \frac{i}{2}(\bar{\epsilon}\gamma_\mu\lambda - \bar{\lambda}\gamma_\mu\epsilon) \\
\delta\sigma &= \frac{1}{2}(\bar{\epsilon}\lambda - \bar{\lambda}\epsilon) \\
\delta\lambda &= -\frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - \epsilon D + i\gamma^\mu\epsilon D_\mu\sigma + \frac{2i}{3}\sigma\gamma^\mu D_\mu\epsilon \\
\delta\bar{\lambda} &= -\frac{1}{2}\gamma^{\mu\nu}\bar{\epsilon}F_{\mu\nu} + \bar{\epsilon}D - i\gamma^\mu\epsilon D_\mu\sigma - \frac{2i}{3}\sigma\gamma^\mu D_\mu\bar{\epsilon} \\
\delta D &= -\frac{i}{2}\bar{\epsilon}\gamma^\mu D_\mu\lambda - \frac{i}{2}D_\mu\bar{\lambda}\gamma^\mu\epsilon + \frac{i}{2}[\bar{\epsilon}\lambda,\sigma] + \frac{i}{2}[\bar{\lambda}\epsilon,\sigma] - \frac{i}{6}D_\mu\bar{\epsilon}\gamma^\mu\lambda - \frac{i}{6}\bar{\lambda}\gamma^\mu D_\mu\epsilon\,.
\end{aligned}
\tag{2.3}
$$

In order for the supersymmetry algebra to close the spinors ϵ, $\bar{\epsilon}$ must satisfy

$$
D_\mu\epsilon = \frac{i}{2r}\gamma_\mu\epsilon\,, \qquad D_\mu\bar{\epsilon} = \frac{i}{2r}\gamma_\mu\bar{\epsilon}\,. \tag{2.4}
$$

Supersymmetric Lagrangians for the vector multiplet can be split into Yang Mills, Chern Simons and Fayet Iliopoulos components

$$
\mathcal{L}_{\text{vec}} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{FI}}\,. \tag{2.5}
$$

The Yang Mills Lagrangian is

$$
\begin{aligned}
\mathcal{L}_{\text{YM}} &= \frac{1}{g^2}\,\mathrm{Tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\sigma D^\mu\sigma + \frac{1}{2}(D+\frac{\sigma}{r})^2 + \frac{i}{2}\bar{\lambda}\gamma^\mu D_\mu\lambda + \frac{i}{2}\bar{\lambda}[\sigma,\lambda] - \frac{1}{4r}\lambda\bar{\lambda}\right] \\
&= \frac{1}{g^2}\delta_\epsilon\delta_{\bar{\epsilon}}\,\mathrm{Tr}[\frac{1}{2}\bar{\lambda}\lambda - 2D\sigma]\,,
\end{aligned}
\tag{2.6}
$$

where r is the radius of S_3 and g is the gauge coupling. In the second line we wrote the Lagrangian as a total superderivative. This is very important, because it means that the YM action can itself play the role of δV in (2.15), with the role of t being assumed by the gauge coupling. By the reasoning of (2.16), this demonstrates that the partition function is independent of the gauge coupling.

The Chern Simons Lagrangian is

$$
\mathcal{L}_{\text{CS}} = \frac{ik}{4\pi}\,\mathrm{Tr}[\epsilon^{\mu\nu\rho}(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho) - \bar{\lambda}\lambda + 2D\sigma]\,, \tag{2.7}
$$

where k is the Chern Simons level, which must be integer to ensure that the action is invariant under large gauge transformations [67]

Finally the Fayet Iliopoulos Lagrangian is

$$\mathcal{L}_{\rm FI} = \frac{i\zeta}{2\pi}\,{\rm Tr}[\frac{\sigma}{r} - D]\,, \tag{2.8}$$

where ζ is the FI parameter

2.1.2 Matter multiplets

An $\mathcal{N} = 2$ chiral (or matter) multiplet has two complex scalars ϕ and F, and two fermions ψ, $\bar{\psi}$, which can transform in any representation of the gauge group. Supersymmetry variations are

$$\begin{aligned}
\delta\phi &= \bar{\epsilon}\psi \\
\delta\bar{\phi} &= \epsilon\bar{\psi} \\
\delta\psi &= i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\sigma\phi + \frac{2iq}{3}\gamma^\mu D_\mu\epsilon\phi + \bar{\epsilon}F \\
\delta\bar{\psi} &= i\gamma^\mu\bar{\epsilon}D_\mu\bar{\phi} + i\bar{\phi}\sigma\bar{\epsilon} + \frac{2iq}{3}\bar{\phi}\gamma^\mu D_\mu\bar{\epsilon} + \epsilon\bar{F} \\
\delta F &= i\epsilon\gamma^\mu D_\mu\psi - i\epsilon\psi\sigma - i\epsilon\phi\lambda + \frac{i}{3}(2q-1)D_\mu\epsilon\gamma^\mu\psi \\
\delta\bar{F} &= i\bar{\epsilon}\gamma^\mu D_\mu\bar{\psi} - i\bar{\epsilon}\bar{\psi}\sigma + i\bar{\epsilon}\bar{\phi}\bar{\lambda} + \frac{i}{3}(2q-1)D_\mu\bar{\epsilon}\gamma^\mu\bar{\psi}\,,
\end{aligned} \tag{2.9}$$

where q is the dimension of the field ϕ, which will usually take the canonical value $q = \frac{1}{2}$.

The supersymmetric Lagrangian for matter multiplets is given by

$$\begin{aligned}
\mathcal{L}_{\rm mat} &= {\rm Tr}\Big[D_\mu\bar{\phi}D^\mu\phi + \bar{\phi}\sigma^2\phi + i\frac{2q-1}{r}\bar{\phi}\sigma\phi + \frac{q(2-q)}{r^2}\bar{\phi}\phi + i\bar{\phi}D\phi \\
&\qquad + \bar{F}F - i\bar{\psi}\gamma^\mu D_\mu\psi + i\bar{\psi}\sigma\psi - \frac{2q-1}{2r}\bar{\psi}\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi\Big] \\
&= \delta_\epsilon\delta_{\bar{\epsilon}}\,{\rm Tr}\big[\psi\bar{\psi} - 2i\bar{\phi}\sigma\phi - \frac{2q-2}{r}\bar{\phi}\phi\big]\,,
\end{aligned} \tag{2.10}$$

where in the second line we have again written the Lagrangian as a total superderivative.

2.1.3 $\mathcal{N} > 2$ supersymmetry

In general we consider theories with $\mathcal{N} > 2$ supersymmetry, which contain $\mathcal{N} = 4$ vector and hypermultiplets. An $\mathcal{N} = 4$ vector multiplet can be built from an $\mathcal{N} = 2$ vector multiplet plus an additional $\mathcal{N} = 2$ chiral multiplet in the adjoint representation with dimension $q = 1$, whose component fields will be distinguished by bold font. The combined Lagrangian $\mathcal{L}_{\rm mat} + \mathcal{L}_{\rm YM}$ has an extended $SU(2)_R \times$

$SU(2)_L$ R symmetry, under which the scalars $(\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}, \sigma)$ and $(\boldsymbol{F}, \bar{\boldsymbol{F}}, D)$ transform as $(\mathbf{3}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{3})$ respectively, while the fermions $(\lambda, \bar{\lambda}, \boldsymbol{\psi}, \bar{\boldsymbol{\psi}})$ transform as $(\mathbf{2}, \mathbf{2})$.

An $\mathcal{N} = 4$ hypermultiplet is made up of a pair of $\mathcal{N} = 2$ chiral multiplets transforming in conjugate representations R and $\bar{R}$. The component fields of the conjugate multiplet will be distinguished by a tilde. The combinations of matter Lagrangians $\mathcal{L}_{\text{mat}R} + \mathcal{L}_{\text{mat}\bar{R}}$ has enhanced $\mathcal{N} = 4$ supersymmetry provided the dimension q takes the canonical value of $\frac{1}{2}$. Under the $SU(2)_R \times SU(2)_L$ R symmetry the scalars $(\phi, \bar{\tilde{\phi}})$ and $(\tilde{\phi}, \bar{\phi})$ transform as $(\mathbf{1}, \mathbf{2})$ and $(\mathbf{2}, \mathbf{1})$ respectively, while the fermions $(\psi, \bar{\psi}, \tilde{\psi}, \bar{\tilde{\psi}})$ transform as $(\mathbf{2}, \mathbf{2})$. Finally, in order to respect also the R symmetry transformations of the vector multiplet fields, one needs an additional superpotential term, that couples the hypermultiplet fields with those of the gauge chiral multiplet.

$$\mathcal{L}_{\text{potential}} = i\sqrt{2}\,\mathrm{Tr}[\tilde{F}\boldsymbol{\phi}\phi + \tilde{\phi}\boldsymbol{F}\phi + \tilde{\phi}\boldsymbol{\phi}F + 2\tilde{\phi}\boldsymbol{\psi}\psi + 2\tilde{\psi}\boldsymbol{\phi}\psi + 2\tilde{\psi}\boldsymbol{\psi}\phi] + c.c. \qquad (2.11)$$

A CS term for an $\mathcal{N} = 4$ vector multiplet has additional quadratic terms for the fields of the gauge chiral multiplet

$$\mathcal{L}_{\text{CS potential}} = \frac{ik}{4\pi}\,\mathrm{Tr}[\boldsymbol{F}\boldsymbol{\phi} + \bar{\boldsymbol{F}}\bar{\boldsymbol{\phi}} + \frac{1}{2}\boldsymbol{\psi}\boldsymbol{\psi} + \frac{1}{2}\bar{\boldsymbol{\psi}}\bar{\boldsymbol{\psi}}] \qquad (2.12)$$

Adding a CS term ((2.7) plus (2.12)) to the Lagrangian of an $\mathcal{N} = 4$ YM-matter theory generally breaks the supersymmetry to $\mathcal{N} = 3$, and the corresponding $SU(2)$ R symmetry acts on the triplets $(\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}, \sigma)$ and $(\boldsymbol{F}, \bar{\boldsymbol{F}}, D)$ simultaneously. For some exceptional theories with specific configurations of CS levels and matter multiplets, a higher degree of supersymmetry can be maintained even in the presence of CS terms [12, 68, 69]. We will encounter examples of such theories when we come to study mirror symmetry in section 2.5.

2.1.4 Quiver theories

The theories we study are quiver gauge theories, whose gauge groups are given by a product

$$G = \prod_a G^{(a)} \qquad (2.13)$$

There are $\mathcal{N} = 4$ vector multiplets for each gauge group factor $G^{(a)}$, and any number of $\mathcal{N} = 4$ matter multiplets which transform in nontrivial representations under a single or a pair of gauge factors. The multiplet content of such a theory can be efficiently encoded by a quiver diagram. In such diagrams, vector multiplets will be denoted by circular 'nodes', which are labelled inside by the corresponding gauge group factor. In the vast majority of instances the gauge factors are $U(N)$ and in these cases the corresponding node will often be labelled just by the rank N. Lines between nodes represent bifundamental hypermultiplets which transform in the $(N_a, \bar{N}_b) \oplus (\bar{N}_a, N_b)$ of $G^{(a)} \times G^{(b)}$. A line between a node and a box labelled by n_f represents n_f fundamental hypermultiplets.

In a few examples we will include additional data with quiver diagrams such as CS levels or FI parameters, or present quiver theories with matter multiplets in other representations. We will make the labelling rules for these clear when we present such diagrams later in the text.

2.2 Localisation

The partition function is given by an (infinite dimensional) path integral over all of the fields

$$Z = \int \mathcal{D}[\phi]\, e^{S} \tag{2.14}$$

This integral is extremely challenging to compute, and for a long time progress could only be made through a weak coupling expansion. A breakthrough came in [6], which developed a new technique, supersymmetric localisation, to study the path integral of $\mathcal{N} = 4$ SYM on S^4 exactly.

The basic idea is to deform the action by a total superderivative $t\delta V$, where the superderivative δ satisfies $\delta^2 = 0$, V is given by some combination of fields and t is an auxiliary parameter. The partition function becomes

$$Z(t) = \int \mathcal{D}[\phi]\, e^{S - t\delta V}\,. \tag{2.15}$$

In fact the expression above is independent of t, since

$$\partial_t Z(t) = \int \mathcal{D}[\phi]\, \delta V e^{S-t\delta V} = \int \mathcal{D}[\phi]\, \delta\big(V e^{S-t\delta V}\big) = 0\,, \tag{2.16}$$

where in the second equality we made use of the fact that the action is invariant under supersymmetry, and that the superderivative δ satisfies $\delta^2 = 0$. In particular, since $Z(t)$ is independent of t, we are free to take $t \to \infty$. Now, if δV has positive definite bosonic part, then taking $t \to \infty$ 'localises' the path integral onto field configurations which set this to zero. By expanding the fields around this locus most of the remaining integrals become Gaussian and can be evaluated exactly. In many cases what we are left with is just a finite dimensional matrix model. This is certainly true for the 3d theories we study here - whose partition functions on S_3 were localised in [7–9] (see also the review [14]), the details of which we now review.

We begin by reviewing the calculation for $\mathcal{N} = 2$ theories and then review how the results can be applied straightforwardly to $\mathcal{N} = 3$ and $\mathcal{N} = 4$ theories.

For $\mathcal{N} = 2$ quiver theories the path integral (2.14) becomes

$$Z = \frac{1}{\mathrm{Vol}\, G} \int \mathcal{D}[\phi] e^{\int d^3x \, \sum_{\mathrm{vec}} (\mathcal{L}_{\mathrm{CS}} + \mathcal{L}_{\mathrm{FI}} - t\mathcal{L}_{\mathrm{YM}}) - \sum_{\mathrm{hyp}} t' \mathcal{L}_{\mathrm{mat}}} \,, \tag{2.17}$$

where the sums are over the vector and chiral multiplets present in the theory, and we have included a normalisation by the volume of the gauge group. As shown in (2.6) and (2.10), $\mathcal{L}_{\mathrm{YM}}$ and $\mathcal{L}_{\mathrm{mat}}$ are total superderivatives, and so by the logic of (2.16) we are free to introduce the auxiliary parameters t, t' for each vector/chiral multiplet without altering the partition function. To perform localisation we take the limit of each of these parameters to infinity and expand the fields around the resulting loci. We do this now separately for the gauge and matter sectors

2.2.1 Gauge sector

Here we consider the contribution to the action from one vector multiplet

$$S_{\mathrm{vec}} = \int d^3x \, (\mathcal{L}_{\mathrm{CS}} + \mathcal{L}_{\mathrm{FI}} - t\mathcal{L}_{\mathrm{YM}}) \,. \tag{2.18}$$

Taking the limit of $t \to \infty$, it is easy to see from (2.6) that the path integrand is suppressed unless

$$F_{\mu\nu} = 0 \,, \qquad D_\mu \sigma = 0 \,, \qquad D + \frac{\sigma}{r} = 0 \,. \tag{2.19}$$

These conditions imply that

$$A_\mu = 0 \,, \qquad \sigma = \sigma_0 \,, \qquad D = -\frac{\sigma_0}{r}, \tag{2.20}$$

where σ_0 is a constant field configuration.

Before proceeding to expand around this locus we should gauge fix A_μ, which will cancel the factor[1]of $\mathrm{Vol}\, G$ in (2.17). This is done by the Fadeev Poppov method [70], which introduces into the path integral a delta function constraint that removes configurations of A_μ that are pure gauge, as well as a factor of the Fadeev Poppov determinant. These can be absorbed by introducing ghost fields b and c,$\bar{c}$ and including a piece in the action

$$S_{\mathrm{ghost}} = \int d^3x \; t(\bar{c}\partial^\mu D_\mu c + bD^\mu A_\mu) \,. \tag{2.21}$$

[1]The gauge fixing procedure removes configurations of A_μ that are pure gauge, namely a gauge transformation of the flat connection $A_\mu = 0$. However there are also gauge transformations which leave the flat connection invariant which form the isotropy group G'. Therefore the gauge fixing procedure does not cancel the factor $(\mathrm{vol}\, G)^{-1}$ entirely, but rather leaves behind a factor of $(\mathrm{vol}\, G')^{-1}$. We neglect this overall normalisation for now.

We now expand/rescale the fields around the locus (2.20)

$$
\begin{gathered}
\sigma = \sigma_0 + \frac{1}{\sqrt{t}}\sigma' \,, \\
D = -\frac{\sigma_0}{r} + \frac{1}{\sqrt{t}} D' \,, \\
A_\mu \to \frac{1}{\sqrt{t}} A_\mu \,, \quad \lambda \to \frac{1}{\sqrt{t}}\lambda \,, \quad c \to \frac{1}{\sqrt{t}} c \,, \quad b \to \frac{1}{\sqrt{t}} b \,.
\end{gathered}
\tag{2.22}
$$

These replacements have a combined unit Jacobian so that the path integral measure is unchanged. Substituting these replacements into (2.18) and (2.21) and taking the limit $t \to \infty$ gives

$$
\begin{aligned}
S_{\text{vec}} + S_{\text{ghost}} = {} & \mathrm{Vol}(S^3)\left(\frac{ik}{2\pi r}\,\mathrm{Tr}(\sigma_0^2) + \frac{i\zeta}{\pi r}\,\mathrm{Tr}(\sigma_0)\right) \\
& + \frac{1}{2}\int d^3x\,\mathrm{Tr}\,\Bigg(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu + \partial_\mu \sigma' \partial^\mu \sigma' - [A_\mu, \sigma_0]^2 + (D' + \frac{\sigma'}{r})^2 \\
& \qquad + i\bar{\lambda}\gamma^\mu \nabla_\mu \lambda + i\bar{\lambda}[\sigma_0, \lambda] - \frac{1}{2r}\lambda\bar{\lambda} + \bar{c}\partial^\mu \partial_\mu c + b\partial^\mu A_\mu\Bigg) \,.
\end{aligned}
\tag{2.23}
$$

We first split A_μ into a pure divergence piece $\partial_\mu \phi$ and a divergenceless piece B_μ

$$
A_\mu = \partial_\mu \phi + B_\mu \,, \qquad \partial^\mu B_\mu = 0 \,. \tag{2.24}
$$

Under this splitting the path integral measure picks up a Jacobian factor

$$
\mathcal{D}A_\mu = \sqrt{\det \Delta^0}\, \mathcal{D}\phi \mathcal{D}B_\mu \,, \tag{2.25}
$$

where Δ^0 is the scalar Laplacian. Now we turn to evaluating the path integrals of (2.23). The integral over b gives a delta function which trivialises the integral over ϕ

$$
\delta(\partial^\mu A_\mu) = \delta(\Delta^0 \phi) = \frac{1}{\det \Delta^0}\delta(\phi) \,. \tag{2.26}
$$

The integral over D' is a simple Gaussian which removes the $(D' + \frac{\sigma'}{r})$ term. Integrals over c,$\bar{c}$ and σ' then give factors of $\det \Delta^0$ and $\frac{1}{\sqrt{\det \Delta^0}}$, so that such factors cancel entirely with those in (2.25) and (2.26). We are left with path integrals over B^μ and λ, and a regular (matrix valued) integral over σ_0. The gauge sector action (2.23) has been reduced to

$$
\begin{aligned}
& i\pi k\,\mathrm{Tr}(\sigma_0^2) + 2i\pi\zeta\,\mathrm{Tr}(\sigma_0) \\
& \quad + \frac{1}{2}\int d^3x\,\mathrm{Tr}\left(B^\mu \Delta^1 B_\mu - [B_\mu, \sigma_0]^2 + i\bar{\lambda}\gamma^\mu \nabla_\mu \lambda + i\bar{\lambda}[\sigma_0, \lambda] - \frac{1}{2}\lambda\bar{\lambda}\right) \,,
\end{aligned}
\tag{2.27}
$$

where we have used $\mathrm{Vol}(S^3) = 2\pi^2 r^3$, set the radius of S^3 to 1 and Δ^1 is the vector laplacian.

To make further progress we need to gauge fix σ_0, so that it takes values in the Cartan subalgebra. This gauge fixing procedure goes as follows. First recall the Cartan decomposition of an adjoint valued scalar

$$\sigma_0 = \sum_{\alpha \neq 0} \sigma_0^\alpha X_\alpha + \sum_i \sigma_0^i h_i \,, \tag{2.28}$$

where σ_0^α and σ_0^i are ordinary scalars, h_i form an orthogonal basis of the Cartan subalgebra, and the X_α are elements of the (1-dimensional) root spaces, defined through

$$[h, X_\alpha] = \alpha(h) X_\alpha \,, \tag{2.29}$$

where h denotes any element of the Cartan subalgebra. The roots are normalised so that

$$\mathrm{Tr}\, X_\alpha X_\beta = \delta_{a+b} \,. \tag{2.30}$$

A suitable Fadeev-Poppov determinant for gauge fixing can then be constructed

$$\Delta^{-1}(\sigma_0) = \int dU \prod_{\alpha \neq 0} \delta(\mathrm{Tr}({}^U\sigma_0 \cdot X_{-\alpha})) \,, \tag{2.31}$$

where U is a gauge transformation, and the delta functions impose that ${}^U\sigma_0$ is in the Cartan subalgebra. Inserting this into an integral over a gauge invariant function $f(\sigma_0)$ gives

$$\begin{aligned}
\int d\sigma_0 f(\sigma_0) &= \int d\sigma_0\, dU f(\sigma_0)\Delta(\sigma_0) \prod_{\alpha \neq 0} \delta(\mathrm{Tr}({}^U\sigma_0 \cdot X_{-\alpha})) \\
&= \mathrm{Vol}(U) \int d\sigma_0 f(\sigma_0)\Delta(\sigma_0) \prod_{\alpha \neq 0} \delta(\mathrm{Tr}(\sigma_0 \cdot X_{-\alpha})) \\
&= \mathrm{Vol}(U) \int d\sigma_0' f(\sigma_0')\Delta(\sigma_0') \,,
\end{aligned} \tag{2.32}$$

where in the second line we have made a gauge transformation $\sigma_0 \to {}^{U^{-1}}\sigma_0$ and used the fact that $f(\sigma_0)$ and $\Delta(\sigma_0)$ are gauge invariant. In the third line σ_0' denotes the Cartan subalgebra. It remains to evaluate the Fadeev Poppov determinant $\Delta(\sigma_0')$. We have

$$\Delta^{-1}(\sigma_0') = \int dU \prod_{\alpha \neq 0} \delta(\mathrm{Tr}((\sigma_0' + [U, \sigma_0']) \cdot X_{-\alpha})) \,. \tag{2.33}$$

Using a Cartan decomposition of U as in (2.28)

$$U = \sum_{\alpha \neq 0} U^\alpha X_\alpha + \sum_i U^i h_i \,, \tag{2.34}$$

along with (2.30) and (2.29), (2.33) becomes proportional to the Vandermonde determinant

$$\begin{aligned}\Delta^{-1}(\sigma_0') &= \mathrm{Vol}(U')\prod_{\alpha\neq 0}\int dU^\alpha \delta(-U^\alpha \alpha(\sigma_0')) \\ &= \mathrm{Vol}(U')\prod_{\alpha>0}\alpha(\sigma_0')^{-2}\,,\end{aligned} \tag{2.35}$$

where U' is the Cartan subalgebra. Finally plugging this back into (2.32) yields the gauge fixed result

$$\int d\sigma_0 f(\sigma_0) = \frac{\mathrm{Vol}(U)}{\mathrm{Vol}(U')}\int d\sigma_0' f(\sigma_0')\prod_{\alpha>0}\alpha(\sigma_0')^2 \tag{2.36}$$

The overall pre factor $\frac{\mathrm{Vol}\,U}{\mathrm{Vol}\,U'}$ will be neglected,[2]and from now on we will omit the prime on σ_0', and take σ_0 to denote the Cartan subalgebra.

Having gauge fixed the integral over σ_0, we return to (2.27) and replace B_μ and λ also by their Cartan decompositions

$$B_\mu = \sum_{\alpha\neq 0} B_\mu^\alpha X_\alpha + \sum_i B_\mu^i h_i\,, \qquad \lambda = \sum_{\alpha\neq 0}\lambda^\alpha X_\alpha + \sum_i \lambda^i h_i\,, \tag{2.37}$$

where B_μ^i, B_μ^α, λ^i and λ^α are now ordinary vector and spinor fields. Then (2.27) becomes

$$\begin{aligned}&i\pi k\,\mathrm{Tr}(\sigma_0^2) + 2i\pi\zeta\,\mathrm{Tr}(\sigma_0) \\ &+ \frac{1}{2}\int d^3x \sum_{\alpha\neq 0} B^{\mu\alpha}\Big(\Delta^1 - \alpha(\sigma_0)^2\Big)B_\mu^{-\alpha} + \bar{\lambda}^\alpha\Big(i\gamma^\mu\nabla_\mu + i\alpha(\sigma_0) - \tfrac{1}{2}\Big)\lambda^{-\alpha} \\ &+ \frac{1}{2}\int d^3x \sum_i \mathrm{Tr}(h_i^2)\Big(B^{\mu i}\Delta^1 B_\mu^i + \bar{\lambda}^i\Big(i\gamma^\mu\nabla_\mu - \tfrac{1}{2}\Big)\lambda^i\Big)\end{aligned} \tag{2.38}$$

The path integrals over the Cartan components B^i and λ^i that appear in the third line give determinants that contribute only an overall numerical factor, which we neglect for now. The path integrals over the root components B_μ^α and λ^α on the other hand give determinants which depend explicitly on σ_0. The partition function for a single vector multiplet is thus reduced to

$$Z_{\mathrm{vec}} \sim \int d\sigma_0 e^{i\pi k\,\mathrm{Tr}(\sigma_0^2) + 2i\pi\zeta\,\mathrm{Tr}(\sigma_0)}\Big(\prod_{\alpha>0}\alpha(\sigma_0)^2\Big)\prod_{\alpha\neq 0}\frac{\det_\lambda\left(i\gamma^\mu\nabla_\mu + i\alpha(\sigma_0) - \frac{1}{2}\right)}{\left(\det_{B_\mu}\left(\Delta^1 - \alpha(\sigma_0)^2\right)\right)^{\frac{1}{2}}}\,. \tag{2.39}$$

As discussed in [72], on S_3 the operator $i\gamma^\mu\nabla_\mu$ has eigenvalues $\pm(n+\frac{1}{2})$ with multiplicities $n(n+1)$ while Δ^1 (restricted to divergenceless vector fields) has eigenvalues $(2n+1)^2$ with multiplicity $2n(n+1)$. The ratio of determinants appearing in (2.39)

[2]For simple Lie groups it can be evaluated by choosing $f(\sigma_0)$ such that the left hand side of (2.36) is Gaussian and can be integrated directly (see e.g. appendix 2 of [71]).

thus evaluates to

$$\prod_{\alpha\neq 0}\frac{\det_\lambda\left(i\gamma^\mu\nabla_\mu+i\alpha(\sigma_0)-\frac{1}{2}\right)}{\left(\det_{B_\mu}\left(\Delta^1-\alpha(\sigma_0)^2\right)\right)^{\frac{1}{2}}}=\prod_{\alpha\neq 0}\prod_{n=1}^{\infty}\frac{(n+i\alpha(\sigma_0))^{n(n+1)}(-n-1+i\alpha(\sigma_0))^{n(n+1)}}{((n+1)^2+\alpha(\sigma_0)^2)^{n(n+2)}}$$
$$=\prod_{\alpha\neq 0}\prod_{n=1}^{\infty}\frac{(n+i\alpha(\sigma_0))^{n+1}}{(n-i\alpha(\sigma_0))^{n-1}}$$
$$=\prod_{\alpha>0}\prod_{n=1}^{\infty}(n^2+\alpha(\sigma_0)^2)^2$$
$$=\left(\prod_{\alpha>0}\pi^{-2}\prod_{n=1}^{\infty}n^4\right)\prod_{\alpha>0}\frac{\sinh^2\pi\alpha(\sigma_0)}{\alpha(\sigma_0)^2}\,. \tag{2.40}$$

The denominator cancels exactly with the Vandermonde determinant in (2.39). The divergent factor appearing in front can be regularised via zeta function regularisation, and will give some contribution to the overall normalisation which we have anyhow not been keeping track of. In fact, the total normalisation can be fixed by studying the pure Chern Simons theory, whose Lagrangian is just (2.7). The corresponding path integral can be reduced to a matrix model by direct integration [73] and comparing this matrix model with the one obtained by localisation allows one to fix the normalisation completely. The result is finally

$$Z_{\text{vec}}=\frac{1}{|\mathcal{W}|}\int d\sigma_0 e^{i\pi k\,\text{Tr}(\sigma_0^2)+2i\pi\zeta\,\text{Tr}(\sigma_0)}\prod_{\alpha>0}4\sinh^2\pi\alpha(\sigma_0)\,, \tag{2.41}$$

where $|\mathcal{W}|$ is the order of the Weyl group, the residual group of discrete gauge symmetries of the Cartan subalgebra.

2.2.2 Matter sector

We now examine the contribution to the matrix model coming from the localisation of the fields of a chiral multiplet, which we take to have canonical dimension $q=\frac{1}{2}$

$$S_{\text{hyp}}=-t'\int d^3x\mathcal{L}_{\text{hyp}}\,. \tag{2.42}$$

The field restrictions (2.22) (with $t\to\infty$) imposed by the gauge sector localisation reduce this to

$$S_{\text{hyp}}=-\int d^3xt'\,\text{Tr}\left(\partial_\mu\bar{\phi}\partial^\mu\phi+\bar{\phi}\sigma_0^2\phi+\frac{3}{4}\bar{\phi}\phi-i\bar{\phi}\sigma_0\phi+\bar{F}F-i\bar{\psi}\gamma^\mu\nabla_\mu\psi+i\bar{\psi}\sigma_0\psi\right)\,. \tag{2.43}$$

In fact all of the terms appearing are already quadratic, so we can proceed by simply setting $t'=-1$. If we were to include additional superpotential terms in the

Lagrangian such as (2.11), a localisation around $t' = \infty$ would immediately remove them, so that these do not alter the partition function.

The path integral over F is a trivial Gaussian that will not contribute anything. In order to analyse the path integrals over ϕ and ψ we decompose the fields as

$$\phi = \prod_\rho \phi^\rho T_\rho \,, \qquad \psi = \prod_\rho \psi^\rho T_\rho \,, \tag{2.44}$$

where ϕ^ρ and ψ^ρ are ordinary scalar and spinor fields, while T_ρ are the generators of the representation, labelled by the corresponding weights, ρ. They satisfy (for σ_0 a representative of the Cartan subalgebra)

$$\operatorname{Tr} T_\rho T_{\rho'} = \delta_{\rho\rho'} \,, \qquad \sigma_0 T_\rho = \rho(\sigma_0) T_\rho \,. \tag{2.45}$$

Plugging this into (2.43) we obtain

$$S_{\text{hyp}} = \int d^3x \sum_\rho \Big(\bar{\phi}^\rho \big(\Delta^0 + \rho(\sigma_0)^2 - i\rho(\sigma_0) + \tfrac{3}{4}\big)\phi^\rho + \bar{\psi}^\rho\big(- i\gamma^\mu \nabla_\mu + i\rho(\sigma_0)\big)\psi^\rho \Big) \tag{2.46}$$

As in the gauge sector, the path integrals give a certain ratio of determinants. Using the fact [7] that the eigenvalues of $i\gamma^\mu\nabla_\mu$ are $\pm(n+\frac{1}{2})$ with multiplicities $n(n+1)$ and the eigenvalues of the scalar laplacian Δ^0 are n^2-1 with multiplicities n^2 we can evaluate the determinants to obtain

$$\begin{aligned}
\prod_\rho \frac{\det_\psi \big(- i\gamma^\mu\nabla_\mu + i\rho(\sigma_0)\big)}{\det_\phi \big(\Delta^0 - i\rho(\sigma_0) + \rho(\sigma_0)^2 + \frac{3}{4}\big)} &= \prod_\rho \prod_{n=1}^{\infty} \frac{(n+\frac{1}{2}+i\rho(\sigma_0))^{n(n+1)}(n+\frac{1}{2}-i\rho(\sigma_0))^{n(n+1)}}{(n^2-\frac{1}{4}-i\rho(\sigma_0)+\rho(\sigma_0)^2)^{n^2}} \\
&= \prod_\rho \prod_{n=1}^{\infty} \left(\frac{n+\frac{1}{2}+i\rho(\sigma_0)}{n-\frac{1}{2}-i\rho(\sigma_0)}\right)^n \\
&= \prod_\rho s_{b=1}\Big(\frac{i}{2} - \rho(\sigma_0)\Big) \,,
\end{aligned} \tag{2.47}$$

where s_b is the double sine function [9].

A standard deformation of an $\mathcal{N}=2$ chiral multiplet is to include a coupling to an additional background abelian vector multiplet. The background fields must satisfy $A_\mu = \lambda = 0$, $\sigma = -rD = \sigma_{background}$ so that the YM Lagrangian for the background fields vanishes. The single remaining scalar $\sigma_{\text{background}}$ acts as a real mass m for the chiral multiplet fields, and will appear in (2.47) as a shift of the double sine function[3]

$$\prod_\rho s_{b=1}\Big(\frac{i}{2}-\rho(\sigma_0)-m\Big) \tag{2.48}$$

2.2.3 Localisation for $\mathcal{N}=4$ multiplets

The quiver theories we consider will feature $\mathcal{N}=4$ multiplets. As discussed in section 2.1.3, an $\mathcal{N}=4$ vector multiplet can be constructed by combining an $\mathcal{N}=2$ vector multiplet with an additional $\mathcal{N}=2$ chiral multiplet in the adjoint representation, with dimension $q=1$. The localisation of this additional chiral multiplet follows in exactly the same way as for a matter chiral multiplet, but since $q=1$ the result turns out to be [8] *c.f.*(2.47)

$$\prod_\rho\prod_{n=1}^{\infty}\left(\frac{n+i\rho(\sigma_0)}{n-i\rho(\sigma_0)}\right)^n . \tag{2.49}$$

Since the weights of the adjoint representation always come in pairs related by a minus sign, (2.49) evaluates to unity, and so the contribution to the matrix model for an $\mathcal{N}=4$ vector multiplet remains exactly (2.41).

An $\mathcal{N}=4$ hypermultiplet is made up of conjugate pairs of $\mathcal{N}=2$ chiral multiplets; for every $\mathcal{N}=2$ chiral multiplet in a representation R there will be another in the conjugate representation $\bar{R}$. If we include mass deformations then in order to preserve $\mathcal{N}=4$ the masses of the conjugate chiral multiplets must be related by a minus sign. Since the weights of a representation and those of its conjugate are also related by a minus sign, the total contribution will be, from (2.48)

$$\prod_\rho s_{b=1}\Big(\frac{i}{2}-\rho(\sigma_0)-m\Big)s_{b=1}\Big(\frac{i}{2}+\rho(\sigma_0)+m\Big)=\prod_\rho\frac{1}{2\cosh\pi(\rho(\sigma_0)+m)} . \tag{2.50}$$

2.3 Matrix models for quiver theories

Notations: We consider quiver theories with nodes of rank N or $2N$ and we use the indices i,j when the label runs over $1,\cdots,N$, and I,J when the label runs over $1,\cdots,2N$. Moreover we define

$$\operatorname{sh}x\equiv 2\sinh\pi x\,,\qquad \operatorname{ch}x\equiv 2\cosh\pi x\,,\qquad \operatorname{th}x\equiv\frac{\operatorname{sh}x}{\operatorname{ch}x}=\tanh\pi x\,. \tag{2.51}$$

In the previous sections we reviewed the localisation computation for vector and hypermultiplets. Here we collect these results along with some facts concerning Lie

[3] Another way of saying this is that the representation of the hypermultiplet has changed from R to $R\oplus\mathbf{1}$ where $\mathbf{1}$ is the fundamental representation of the background $U(1)$ gauge group. The localisation calculation is unchanged but the weights are altered from ρ to $\rho+m$.

algebras to allow one to easily associate a matrix model to a quiver gauge theory.

For a quiver theory with gauge group (2.13), an $\mathcal{N}=4$ vector multiplets for each gauge group factor $G^{(a)}$ (with CS levels $k^{(a)}$ and FI parameters $\zeta^{(a)}$) and arbitrary number of $\mathcal{N}=4$ hypermultiplets, the matrix model is

$$Z=\frac{1}{|\mathcal{W}|}\int\prod_{a}d\sigma_0^{(a)}e^{i\pi k^{(a)}\,\mathrm{Tr}((\sigma_0^{(a)})^2)+2i\pi\zeta^{(a)}\,\mathrm{Tr}(\sigma_0^{(a)})}\prod_{\alpha>0}\mathrm{sh}^2\,\alpha(\sigma_0^{(a)}\prod_{\mathrm{hyp}}\prod_{\rho_{\mathrm{hyp}}}\frac{1}{\mathrm{ch}\,\rho_{\mathrm{hyp}}+m_{\mathrm{hyp}}}\,. \tag{2.52}$$

The integral over the Cartan subalgebra $\sigma_0^{(a)}$ of each gauge group factor can be decomposed into N eigenvalue integrals, where N is the rank of the Cartan subalgebra. The roots α will then be given by linear functions of the eigenvalues. $\prod_{\mathrm{hyp}}$ denotes a product over the hypermultiplets, which have a masses m_{hyp}, while ρ_{hyp} denote the weights of the corresponding representation.

In the quiver theories we study the gauge group factors $G^{(a)}$ will always be unitary or symplectic, and the hypermultiplets will transform in fundamental, antisymmetric or bifundamental representations of the gauge group factor(s). We summarise the relevant data for these groups and representations below.

For unitary $U(N)$ groups we have

$$\begin{array}{rrll} \text{Eigenvalues:} & \lambda_i\,, & & i=1,\dots,N\,,\\ \text{Weyl group order:} & |\mathcal{W}| & =N!\,, & \\ \text{Roots:} & \alpha & =\lambda_i-\lambda_j\,, & i\neq j\\ \text{Fundamental weights:} & \rho_{\mathrm{fund}} & =\lambda_i\,, & i=1,\dots,N\,,\\ \text{Antisymmetric weights:} & \rho_{\mathrm{asym}} & =\lambda_i+\lambda_j, & i<j\,. \end{array} \tag{2.53}$$

For symplectic $Sp(2N)$ groups we have

$$\begin{array}{rrll} \text{Eigenvalues:} & \pm\lambda_i\,, & & i=1,\dots,N\,,\\ \text{Weyl group order:} & |\mathcal{W}| & =2^N N!\,, & \\ \text{Roots:} & \alpha & =\begin{cases}\pm\lambda_i\pm\lambda_j, & i<j\\ \pm2\lambda_i & i=1,\dots,N\end{cases} & \\ \text{Fundamental weights:} & \rho_{\mathrm{fund}} & =\pm\lambda_i\,, & i=1,\dots,N\,,\\ \text{Antisymmetric weights:} & \rho_{\mathrm{asym}} & =\begin{cases}\lambda_i-\lambda_j, & i=1,\dots,N\,,\quad j=1,\dots,N\\ \pm(\lambda_i+\lambda_j) & i<j\,.\end{cases} & \end{array} \tag{2.54}$$

Finally the bifundamental representations of two gauge group factors have weights

$$\begin{aligned}
U(N)\times U(M) \quad &: \quad \rho_{\text{bifund}} = \lambda_i - \tilde{\lambda}_j, \qquad i=1,\dots,N, \qquad j=1,\dots,M \quad , \\
Sp(2N)\times U(2N) \quad &: \quad \rho_{\text{bifund}} = \pm\lambda_i - \tilde{\lambda}_J, \qquad i=1,\dots,N, \qquad J=1,\dots,2N \quad , \\
Sp(2N)\times Sp(2N) \quad &: \quad \rho_{\text{bifund}} = \pm\lambda_i \pm \tilde{\lambda}_j, \qquad i=1,\dots,N, \qquad j=1,\dots,N \quad ,
\end{aligned} \tag{2.55}$$

where λ are the eigenvalues associated to the Cartan of the left group and $\tilde{\lambda}$ those of the right.

2.4 From matrix models to free fermions

We now discuss how the matrix models corresponding to partition functions of certain families of $\mathcal{N} \geqslant 3$ theories may be mapped onto partition functions of 1d free fermions, an approach that was pioneered for $\hat{A}$ quivers in [14], and which we adapted for $\hat{D}$ quivers and certain linear quivers in [2].

2.4.1 $\hat{A}$ quivers

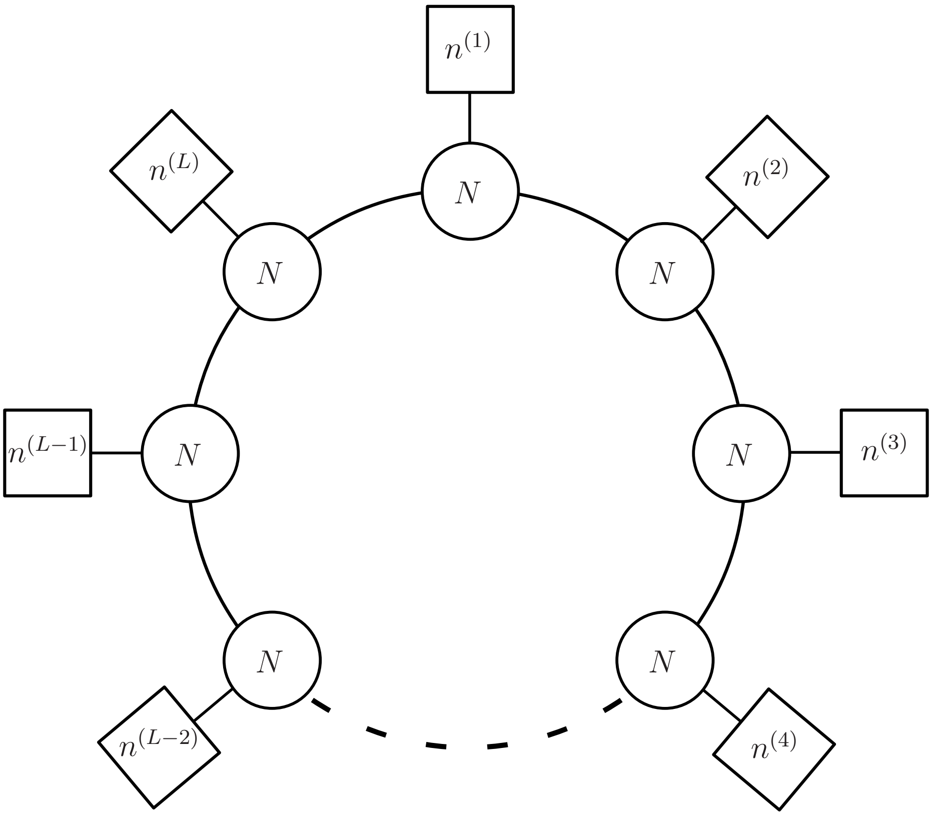

Figure 2.1: The general $\hat{A}_{L-1}$-quiver. $n^{(a)}$ denotes the number of fundamental hypermultiplets at the corresponding $U(N)$ node.

As a first example we review the theories considered in [14], namely $\mathcal{N} \geqslant 3$ $\hat{A}$ quiver theories, who's circular structure makes them most readily amenable to a Fermi gas analysis. We generalise slightly by the inclusion of FI and mass parameters, but this does not introduce any technical difficulties compared with [14]. Concretely, the $\hat{A}_{L-1}$ quiver theories we consider have gauge group $U(N)^L$ with a vector multiplet and arbitrary number of fundamental hypermultiplets for each gauge group factor, and bifundamental hypermultiplets connecting them in a circular fashion, as in Figure 2.1. In addition we allow for arbitrary FI and CS terms for the fields of each vector multiplet, and arbitrary masses for the hypermultiplets. Following the rules in section 2.3, the matrix model for these theories is

$$Z(N) = \frac{1}{N!^L} \int \prod_{a=1}^{L} d^N \lambda^{(a)} \prod_{i=1}^{N} F^{(a)}(\lambda_i^{(a)}) \frac{\prod_{i<j} \mathrm{sh}^2(\lambda_i^{(a)} - \lambda_j^{(a)})}{\prod_{i,j} \mathrm{ch}(\lambda_i^{(a)} - \lambda_j^{(a+1)} + m^{(a)})}, \tag{2.56}$$

where $\lambda_i^{(a)}$, $i = 1, \cdots, N$ are the eigenvalues of the a^{th} node (with the identification $\lambda^{(L+1)} \equiv \lambda^{(1)}$) and $m^{(a)}$ are the bifundamental masses. The factor $\prod_i F^{(a)}(\lambda_i^{(a)})$ represents all terms which depend only on single eigenvalues and includes contributions from the CS term with level $k^{(a)}$, the FI term with parameter $\zeta^{(a)}$ and $n^{(a)}$ fundamental hypermultiplets with individual masses $\mu_\alpha^{(a)}$, $\alpha = 1, \cdots, n^{(a)}$

$$F^{(a)}(\lambda) = e^{\pi i k^{(a)} \lambda^2} e^{2\pi i \zeta^{(a)} \lambda} \prod_{\alpha=1}^{n^{(a)}} \frac{1}{\mathrm{ch}(\lambda + \mu_\alpha^{(a)})}. \tag{2.57}$$

Using the Cauchy determinant identity

$$\frac{\prod_{i<j} \mathrm{sh}(\lambda_i - \lambda_j) \,\mathrm{sh}(\tilde{\lambda}_i - \tilde{\lambda}_j)}{\prod_{i,j} \mathrm{ch}(\lambda_i - \tilde{\lambda}_j)} = \sum_{\sigma \in S_N} (-1)^\sigma \prod_{i=1}^{N} \frac{1}{\mathrm{ch}(\lambda_i - \tilde{\lambda}_{\sigma(i)})}, \tag{2.58}$$

the partition function can be re-expressed as a sum over L permutations

$$Z(N) = \frac{1}{N!^L} \sum_{\sigma^{(a)} \in S_N} (-1)^{\sum_{a=1}^{L} \sigma^{(a)}} \int \prod_{a=1}^{L} d^N \lambda^{(a)} \prod_{i=1}^{N} \frac{F^{(a)}(\lambda_i^{(a)})}{\mathrm{ch}(\lambda_i^{(a)} - \lambda_{\sigma^{(a)}(i)}^{(a+1)} + m^{(a)})}. \tag{2.59}$$

By relabelling the eigenvalues one can factor out all but one of the permutations, picking up an overall factor of $N!^{L-1}$. This gives

$$\begin{aligned} Z(N) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma \int \prod_{a=1}^{L} d^N \lambda^{(a)} \prod_{i=1}^{N} \left(\prod_{a=1}^{L-1} \frac{F^{(a)}(\lambda_i^{(a)})}{\mathrm{ch}(\lambda_i^{(a)} - \lambda_i^{(a+1)} + m^{(a)})} \right) \\ \times \frac{F^{(L)}(\lambda_i^{(L)})}{\mathrm{ch}(\lambda_i^{(L)} - \lambda_{\sigma(i)}^{(1)} + m^{(L)})}. \end{aligned} \tag{2.60}$$

This integrand is a series of kernels of pairs of specific eigenvalues of successive nodes, ultimately coupling each $\lambda_i^{(1)}$ with $\lambda_{\sigma(i)}^{(1)}$. This can be encoded graphically by the diagram

$$\left\{\lambda^{(1)}\right\} \to \left\{\lambda^{(2)}\right\} \to \cdots \to \left\{\lambda^{(L)}\right\} \xrightarrow{\sigma} \left\{\lambda^{(1)}\right\} . \tag{2.61}$$

One can express these kernels in terms of canonical position and momentum operators $\hat{q}$, $\hat{p}$, which satisfy $[\hat{q}, \hat{p}] = i\hbar$. Henceforth we will suppress the hats on operators. Taking λ to be position eigenvalues, we have

$$F(\lambda)\delta(\lambda' - \lambda) = \langle \lambda' | \, F(q) \, |\lambda\rangle \, , \quad \frac{1}{\operatorname{ch}(\lambda - \lambda')} = \langle \lambda | \, \frac{1}{\operatorname{ch} p} \, |\lambda'\rangle \, , \quad e^{2\pi i m p} \, |\lambda\rangle = |\lambda - m\rangle \, . \tag{2.62}$$

Additional important identities are

$$e^{\pi i n q^2} f(p) e^{-\pi i n q^2} = f(p + nq) \, , \qquad e^{\pi i n p^2} f(q) e^{-\pi i n p^2} = f(q - np) \, . \tag{2.63}$$

Here we have used the standard relation between the position and momentum bases $|p\rangle = \int \frac{d\lambda}{\sqrt{2\pi\hbar}} \, e^{\frac{ip\lambda}{\hbar}} \, |\lambda\rangle$ and we have $\langle \lambda_1 | \, \lambda_2 \rangle = \delta(\lambda_1 - \lambda_2)$ and $\langle p_1 | \, p_2 \rangle = \delta(p_1 - p_2)$. We choose to normalise the momentum operator p such that $\hbar = \frac{1}{2\pi}$.

This allows one to write the integrand of (2.60) as

$$\langle \lambda_i^{(1)} | \, F^{(1)}(q) \frac{e^{2\pi i m^{(1)} p}}{\operatorname{ch} p} \, |\lambda_i^{(2)}\rangle \langle \lambda_i^{(2)} | \, F^{(2)}(q) \frac{e^{2\pi i m^{(2)} p}}{\operatorname{ch} p} \, |\lambda_i^{(3)}\rangle \cdots \langle \lambda_i^{(L)} | \, F^{(L)}(q) \frac{e^{2\pi i m^{(L)} p}}{\operatorname{ch} p} \, |\lambda_{\sigma(i)}^{(1)}\rangle \, , \tag{2.64}$$

and we obtain the final expression for $Z(N)$,

$$Z(N) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^{\sigma} \int d^N \lambda \prod_{i=1}^{N} \langle \lambda_i | \, \rho \, |\lambda_{\sigma(i)}\rangle \tag{2.65}$$

with

$$\rho = \prod_{a=1}^{L} F^{(a)}(q) \frac{e^{2\pi i m^{(a)} p}}{\operatorname{ch} p} \, . \tag{2.66}$$

This expression coincides with the partition function of N non-interacting fermions[4] living on an infinite line, where λ_i is the position of the i^{th} fermion on the line. ρ plays the role of the density operator, related to the Hamiltonian H through $\rho = e^{-H}$.

This concludes the mapping of this particular family of matrix models to a Fermi gas partition function. Of course, there is still work to be done in computing the quantity $Z(N)$. Indeed, (2.65) is completely determined by the spectrum of ρ, and we turn to studying this spectral problem in detail in section 2.6. For now, we turn

[4]The fermionic statistics is understood from the antisymmetrisation over permutations of the positions λ_i.

our attention to other families of matrix models, namely those corresponding to partition functions of $\hat{D}$ and certain linear quiver theories.

2.4.2 $\hat{D}$ quivers

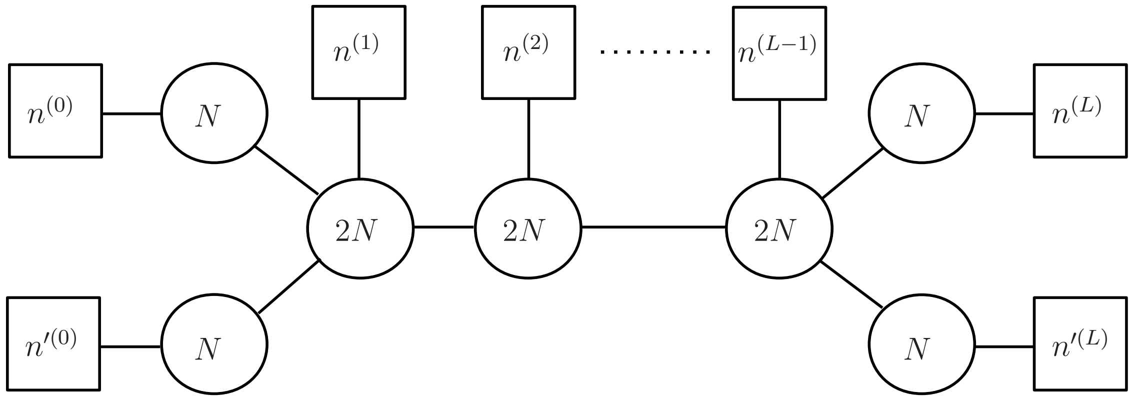

Figure 2.2: The general $\hat{D}_{L+2}$-quiver with arbitrary fundamental matter.

Here we study $\mathcal{N} = 3$ $\hat{D}_{L+2}$ quiver theories, which have gauge group $U(2N)^{L-1} \times U(N)^4$, and quiver structure as in Figure 2.2. Bifundamental hypermultiplets connect the $U(2N)$ gauge nodes in a linear chain, and each terminal $U(2N)$ node with two of the $U(N)$ nodes. We allow for arbitrary CS and FI parameters, as well as arbitrary numbers of fundamental hypermultiplets for each gauge node, and arbitrary masses for all of the hypermultiplets. Although the Fermi gas formalism works in the general case described, the intermediate expressions become very turgid, and so we will work out the full details for a simple example, which will nevertheless involve all of the technical details needed to treat the general case. We describe the straight forward generalisation to longer quivers in section 2.4.4.

$\hat{D}_4$ example

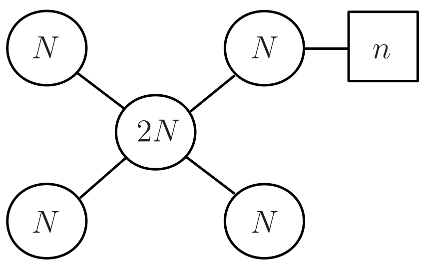

Figure 2.3: A $\hat{D}_4$-quiver with n fundamental hypermultiplets on one $U(N)$ node

Here we consider a $\hat{D}_4$-quiver theory with n fundamental hypermultiplets attached to a single $U(N)$ node, and set all mass, FI and CS parameters to zero.

The quiver diagram for this theory is shown in Figure 2.3. Our aim is to find a suitable density operator that repackages the matrix model for this theory into an expression like (2.65). In order to do this it is useful to collect the eigenvalues of pairs of terminal $U(N)$ nodes into a single set of $2N$ eigenvalues. Anticipating this, we label the eigenvalues of one pair of $U(N)$ nodes respectively by[5] $\lambda_i^{(0)}$ and $\lambda_{N+i}^{(0)}$, $i = 1, \cdots, N$, and the other pair by $\lambda_i^{(2)}$, $\lambda_{N+i}^{(2)}$. The eigenvalues of the $U(2N)$ node are labelled $\lambda_I^{(1)}$, $I = 1, \cdots, 2N$. Following the rules from section 2.3, the matrix model for this theory is given by

$$Z(N) = \frac{1}{N!^4(2N)!} \int \prod_{a=0}^{2} d^{2N}\lambda^{(a)} \prod_{I<J} \mathrm{sh}^2\left(\lambda_I^{(1)} - \lambda_J^{(1)}\right)$$
$$\frac{\prod_{i<j} \mathrm{sh}^2\left(\lambda_i^{(0)} - \lambda_j^{(0)}\right) \mathrm{sh}^2\left(\lambda_{N+i}^{(0)} - \lambda_{N+j}^{(0)}\right) \mathrm{sh}^2\left(\lambda_i^{(2)} - \lambda_j^{(2)}\right) \mathrm{sh}^2\left(\lambda_{N+i}^{(2)} - \lambda_{N+j}^{(2)}\right)}{\prod_{i,J} \mathrm{ch}\left(\lambda_i^{(0)} - \lambda_J^{(1)}\right) \mathrm{ch}\left(\lambda_{N+i}^{(0)} - \lambda_J^{(1)}\right) \mathrm{ch}\left(\lambda_i^{(2)} - \lambda_J^{(1)}\right) \mathrm{ch}\left(\lambda_{N+i}^{(2)} - \lambda_J^{(1)}\right) \prod_i \mathrm{ch}^n \lambda_i^{(2)}} . \tag{2.67}$$

Including artificially the factors $\prod_{i,j} \mathrm{sh}\left(\lambda_i^{(0)} - \lambda_{N+j}^{(0)}\right) \prod_{i,j} \mathrm{sh}\left(\lambda_i^{(2)} - \lambda_{N+j}^{(2)}\right)$ in the numerator and denominator, one can use the Cauchy identity (2.58), as well as a modified version

$$\frac{\prod_{i<j} \mathrm{sh}(\lambda_i - \lambda_j)\, \mathrm{sh}(\tilde{\lambda}_i - \tilde{\lambda}_j)}{\prod_{i,j} \mathrm{sh}(\lambda_i - \tilde{\lambda}_j)} = (-1)^{\frac{N(N-1)}{2}} \sum_{\sigma \in S_N} (-1)^\sigma \prod_{i=1}^{N} \frac{1}{\mathrm{sh}(\lambda_i - \tilde{\lambda}_{\sigma(i)})} , \tag{2.68}$$

to re-expressed $Z(N)$ as

$$Z(N) = \frac{1}{N!^4(2N)!} \sum_{\substack{\sigma^{(a)} \in S_N \\ \tau^{(a)} \in S_{2N}}} (-1)^{\sigma^{(0)}+\sigma^{(2)}+\tau^{(0)}+\tau^{(2)}} \int \prod_{a=0}^{2} d^{2N}\lambda^{(a)} \frac{1}{\prod_i \mathrm{ch}^n \lambda_i^{(2)}}$$
$$\prod_{i=1}^{N} \frac{1}{\mathrm{sh}\left(\lambda_i^{(0)} - \lambda_{N+\sigma^{(0)}(i)}^{(0)}\right)} \frac{1}{\mathrm{sh}\left(\lambda_i^{(2)} - \lambda_{N+\sigma^{(2)}(i)}^{(2)}\right)} \prod_{I=1}^{2N} \frac{1}{\mathrm{ch}\left(\lambda_I^{(0)} - \lambda_{\tau^{(0)}(I)}^{(1)}\right)} \frac{1}{\mathrm{ch}\left(\lambda_I^{(1)} - \lambda_{\tau^{(2)}(I)}^{(2)}\right)} . \tag{2.69}$$

Successive relabellings of the indices allow us to remove the sum over $\sigma^{(0)}$, $\sigma^{(2)}$ and $\tau^{(0)}$ and compensate for it by an overall factor of $N!^2(2N)!$

$$Z(N) = \frac{1}{N!^2} \sum_{\tau \in S_{2N}} (-1)^\tau \int \prod_{a=0}^{2} d^{2N}\lambda^{(a)} \prod_{i=1}^{N} \frac{1}{\mathrm{ch}^n \lambda_i^{(2)}}$$
$$\prod_{i=1}^{N} \frac{1}{\mathrm{sh}\left(\lambda_i^{(0)} - \lambda_{N+i}^{(0)}\right)} \frac{1}{\mathrm{sh}\left(\lambda_i^{(2)} - \lambda_{N+i}^{(2)}\right)} \prod_{I=1}^{2N} \frac{1}{\mathrm{ch}\left(\lambda_I^{(0)} - \lambda_I^{(1)}\right)} \frac{1}{\mathrm{ch}\left(\lambda_I^{(1)} - \lambda_{\tau(I)}^{(2)}\right)} . \tag{2.70}$$

As in the case of the $\hat{A}$-quivers (2.66), we would like to write this as the successive

[5]This is a slight abuse of our earlier notation where superscripts distinguished gauge group factors.

interaction between pairs of eigenvalues. Defining the reflection permutation R by

$$R(i) = N + i\,, \qquad R(N+i) = i\,, \tag{2.71}$$

the integrand of the matrix model can be viewed as a series of kernels pairing eigenvalues of adjacent nodes in a chain that goes back and forth along the quiver, according to the diagram (*c.f.*, (2.61))

$$_R\circlearrowright \left\{\lambda^{(0)}\right\} \rightleftarrows \left\{\lambda^{(1)}\right\} \underset{\tau^{-1}}{\overset{\tau}{\rightleftarrows}} \left\{\lambda^{(2)}\right\} \circlearrowleft_R\,. \tag{2.72}$$

Traversing the quiver back and forth we end up with the composite permutation

$$R\tau^{-1}R\tau\,, \tag{2.73}$$

so we can write the partition function in terms of a kernel relating $\lambda^{(1)}_I$ and $\lambda^{(1)}_{R\tau^{-1}R\tau(I)}$. Note however that another eigenvalue of the central node $\lambda^{(1)}_{\tau^{-1}R\tau(I)}$ is integrated over to get this kernel. So for each permutation τ we need to choose half of the eigenvalues of $\lambda^{(1)}$ on which the kernel acts. Let us call the set of N indices of those eigenvalues $\mathcal{K}(\tau)$. It is chosen to be closed under the composite permutation $R\tau^{-1}R\tau$ and such that R takes this set to its complement $R(\mathcal{K}(\tau)) = \overline{\mathcal{K}(\tau)}$. The partition function can be expressed in the following way

$$\begin{aligned} Z(N) = \frac{1}{N!^2} \sum_{\tau\in S_{2N}} (-1)^\tau \int \prod_{a=0}^{2} d^{2N}\lambda^{(a)} \prod_{i=1}^{N} \frac{1}{\mathrm{ch}^n\,\lambda^{(2)}_i} \prod_{k\in\mathcal{K}(\tau)} \frac{1}{\mathrm{ch}\left(\lambda^{(1)}_k - \lambda^{(2)}_{\tau(k)}\right)} \\ \times \frac{(-1)^{s(\tau(k))}}{\mathrm{sh}\left(\lambda^{(2)}_{\tau(k)} - \lambda^{(2)}_{R\tau(k)}\right)} \frac{1}{\mathrm{ch}\left(\lambda^{(2)}_{R\tau(k)} - \lambda^{(1)}_{\tau^{-1}R\tau(k)}\right)} \frac{1}{\mathrm{ch}\left(\lambda^{(1)}_{\tau^{-1}R\tau(k)} - \lambda^{(0)}_{\tau^{-1}R\tau(k)}\right)} \\ \times \frac{(-1)^{s(\tau^{-1}R\tau(k))}}{\mathrm{sh}\left(\lambda^{(0)}_{\tau^{-1}R\tau(k)} - \lambda^{(0)}_{R\tau^{-1}R\tau(k)}\right)} \frac{1}{\mathrm{ch}\left(\lambda^{(0)}_{R\tau^{-1}R\tau(k)} - \lambda^{(1)}_{R\tau^{-1}R\tau(k)}\right)}\,, \end{aligned} \tag{2.74}$$

where

$$s(k) = \begin{cases} 0\,, & k = 1,\cdots,N\,, \\ 1\,, & k = N+1,\cdots,2N\,. \end{cases} \tag{2.75}$$

To be able to write the partition function in terms of a density operator we need to include the contribution from the fundamental hypermultiplets into the product over k in (2.74). However, the fundamental hypermultiplets couple only to the eigenvalues $\lambda^{(2)}_i$ with $i = 1,\cdots N$, which depending on τ is either $\lambda^{(2)}_{\tau(k)}$ *or* $\lambda^{(2)}_{R\tau(k)}$, but not both. These two options happen with equal probability for each combined permutation $R\tau^{-1}R\tau$, so we can write it as the sum (normalised by $1/2^N$)[6]

[6]In more detail, note that $R\tau^{-1}R\tau$ remains the same if one multiplies τ on the left by any

$$Z(N) = \frac{1}{2^N N!^2} \sum_{\tau\in S_{2N}} (-1)^\tau \int \prod_{a=0}^{2} d^{2N}\lambda^{(a)} \prod_{k\in\mathcal{K}(\tau)} (-1)^{s(k)+s(\tau(k))+1}$$

$$\prod_{k\in\mathcal{K}(\tau)} \Bigg[\frac{1}{\mathrm{ch}\left(\lambda_k^{(1)} - \lambda_{\tau(k)}^{(2)}\right)} \frac{1}{\mathrm{sh}\left(\lambda_{\tau(k)}^{(2)} - \lambda_{R\tau(k)}^{(2)}\right)} \frac{1}{\mathrm{ch}\left(\lambda_{R\tau(k)}^{(2)} - \lambda_{\tau^{-1}R\tau(k)}^{(1)}\right)}$$

$$+ \frac{1}{\mathrm{ch}\left(\lambda_k^{(1)} - \lambda_{R\tau(k)}^{(2)}\right)} \frac{1}{\mathrm{sh}\left(\lambda_{R\tau(k)}^{(2)} - \lambda_{\tau(k)}^{(2)}\right)} \frac{1}{\mathrm{ch}\left(\lambda_{\tau(k)}^{(2)} - \lambda_{\tau^{-1}R\tau(k)}^{(1)}\right)} \Bigg] \frac{1}{\mathrm{ch}^n \lambda_{\tau(k)}^{(2)}}$$

$$\times \frac{1}{\mathrm{ch}\left(\lambda_{\tau^{-1}R\tau(k)}^{(1)} - \lambda_{\tau^{-1}R\tau(k)}^{(0)}\right)} \frac{1}{\mathrm{sh}\left(\lambda_{\tau^{-1}R\tau(k)}^{(0)} - \lambda_{R\tau^{-1}R\tau(k)}^{(0)}\right)} \frac{1}{\mathrm{ch}\left(\lambda_{R\tau^{-1}R\tau(k)}^{(0)} - \lambda_{R\tau^{-1}R\tau(k)}^{(1)}\right)} , \tag{2.76}$$

where we have used $\prod_{k\in\mathcal{K}(\tau)}(-1)^{s(\tau^{-1}R\tau(k))} = \prod_{k\in\mathcal{K}(\tau)}(-1)^{s(k)+1}$. This expression can now be recast as

$$Z(N) = \frac{1}{2^{2N} N!^2} \sum_{\tau\in S_{2N}} (-1)^\tau \int d^N\lambda \prod_{k\in\mathcal{K}(\tau)} (-1)^{s(k)+s(\tau(k))} \rho(\lambda_k, \lambda_{R\tau^{-1}R\tau(k)}) , \tag{2.77}$$

with

$$\rho(\lambda,\lambda') = 2 \int \prod_{a=1}^{5} d\lambda_a \frac{-1}{\mathrm{ch}(\lambda-\lambda_1)} \left(\frac{1}{\mathrm{ch}^n \lambda_1} \frac{1}{\mathrm{sh}(\lambda_1-\lambda_2)} + \frac{1}{\mathrm{sh}(\lambda_1-\lambda_2)} \frac{1}{\mathrm{ch}^n \lambda_2} \right)$$
$$\frac{1}{\mathrm{ch}(\lambda_2-\lambda_3)} \frac{1}{\mathrm{ch}(\lambda_3-\lambda_4)} \frac{1}{\mathrm{sh}(\lambda_4-\lambda_5)} \frac{1}{\mathrm{ch}(\lambda_5-\lambda')} , \tag{2.78}$$

where we chose the normalisation factor for convenience.

The kernel ρ defines a density operator through the relation $\rho(\lambda_1,\lambda_2) = \langle\lambda_1|\,\rho\,|\lambda_2\rangle$, which has a representation in terms of canonical position and momentum operators (2.62)

$$\rho = \frac{1}{2}\frac{1}{\mathrm{ch}\, p} \left(\frac{1}{\mathrm{ch}^n q} \frac{\mathrm{sh}\, p}{\mathrm{ch}\, p} + \frac{\mathrm{sh}\, p}{\mathrm{ch}\, p} \frac{1}{\mathrm{ch}^n q} \right) \frac{\mathrm{sh}\, p}{\mathrm{ch}^4 p} , \tag{2.79}$$

where in addition to (2.62) we have used

$$\frac{1}{\mathrm{sh}(\lambda-\lambda')} = -\frac{i}{2} \langle\lambda|\, \frac{\mathrm{sh}\, p}{\mathrm{ch}\, p}\, |\lambda'\rangle . \tag{2.80}$$

To make further progress, we need to study the combinatorics of the composite permutations $R\tau^{-1}R\tau$. We relegate these additional technical calculations to appendix A and provide the final simplified result, which involves only a sum over

combinations of two-cycles appearing in R. For a given τ this generates a set of 2^N terms, half with $\tau(k)$ in $\{1,\cdots,N\}$ and half in the compliment.

permutations of S_N

$$Z(N) = \frac{1}{N!} \sum_{\sigma \in S_N} \frac{(-1)^\sigma}{2^{n_\sigma}} \int \prod_{i=1}^{N} d\lambda_i \prod_{i=1}^{N} \rho(\lambda_i, \lambda_{\sigma(i)}) \,, \tag{2.81}$$

where n_σ is the number of cycles in σ.

Because of the factor $1/2^{n_\sigma}$, (2.81) cannot be interpreted directly as the partition function of N non-interacting fermions. When all FI and mass parameters are turned off, we find that it can be understood as resulting from a projection onto half of the states of a fermionic system. We show in appendix B that the density operator ρ commutes with the reflection operator $\hat{R}$, defined by

$$\hat{R} \, |\lambda\rangle = |-\lambda\rangle \; . \tag{2.82}$$

Consequently, the Hilbert space can be split into even and odd eigenstates. Furthermore, the spectra of even and odd eigenstates are identical, allowing us to rewrite $Z(N)$, using the projector $\frac{1+\hat{R}}{2}$, as

$$Z(N) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma \int \prod_{i=1}^{N} d\lambda_i \prod_{i=1}^{N} \langle \lambda_i | \, \rho \Big(\frac{1+\hat{R}}{2} \Big) \, |\lambda_{\sigma(i)}\rangle \; . \tag{2.83}$$

This can be readily interpreted as the partition function of N non-interacting fermions at positions $|\lambda_i|$ on a half-line with a Hamiltonian $H = -\log \rho$ where the operator $\frac{1+\hat{R}}{2}$ is responsible for the projection onto particle states with even wavefunction on the line, or equivalently particle states on a half-line with Neumann boundary conditions.

Likewise we can use the projector $\frac{1-\hat{R}}{2}$ to express the partition function in terms of the odd states

$$Z(N) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma \int \prod_{i=1}^{N} d\lambda_i \prod_{i=1}^{N} \langle \lambda_i | \, \rho \Big(\frac{1-\hat{R}}{2} \Big) \, |\lambda_{\sigma(i)}\rangle \; . \tag{2.84}$$

In this case we would interpret $Z(N)$ as the partition function of N non-interacting fermions on a half-line with Dirichlet boundary condition at the origin.

We did not find a free fermion interpretation of the partition function (2.81) for the cases with non-vanishing masses and FI parameters.

2.4.3 Linear $U(2N)$-$Sp(2N)$ quivers

The third class of quiver theories we study comprise again a linear chain of $U(2N)$ gauge nodes connected with bifundamental hypermultiplets. Rather than coupling

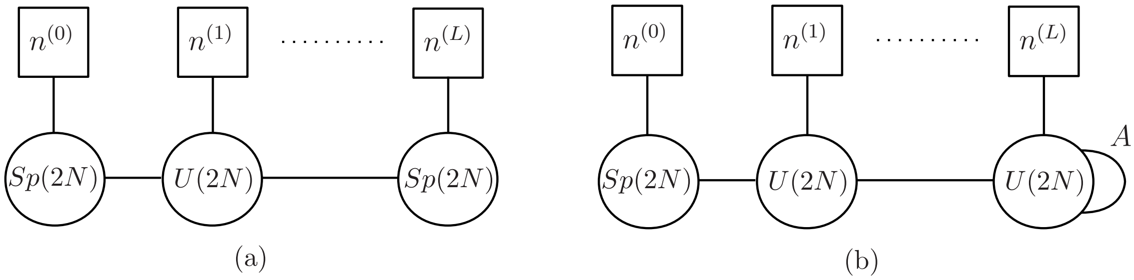

Figure 2.4: Examples of linear quivers that are mirror dual to $\hat{D}$-quivers

to $U(N)$ gauge nodes each terminal $U(2N)$ node is coupled with an antisymmetric hypermultiplet, or is replaced with an $Sp(2N)$ node. These theories are of specific interest as they are known to be mirror dual to $\hat{D}$ quivers. We refer to these theories as *linear* quivers. Examples of such quivers are presented in Figure 2.4. We present the full detail of the free Fermi formalism again for a simple but representative example which contains all of the technical details needed to treat the general case, as will be explained in section 2.4.4.

U-SP example

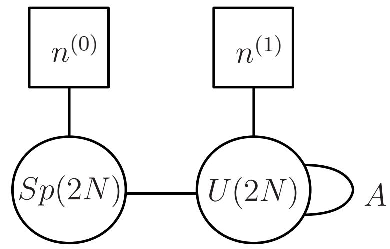

Figure 2.5: A linear quiver with an $Sp(2N)$ node at one end and an antisymmetric hypermultiplet at the other.

Here we consider the $U(2N) \times Sp(2N)$ quiver theory depicted in Figure 2.5. The $Sp(2N)$ node has eigenvalues labelled $\lambda_i^{(0)}$, $i = 1, \cdots, N$ and $n^{(0)}$ fundamental hypermultiplets. The $U(2N)$ node has eigenvalues $\lambda_I^{(1)}$, $I = 1, \cdots, 2N$, an antisymmetric hypermultiplet as well as $n^{(1)}$ fundamental hypermultiplets. We set all mass, CS and FI parameters to zero. Following the rules in section 2.3, the matrix model is given by

$$
\begin{aligned}
Z(N) = \frac{1}{2^N N!(2N)!} \int d^N\lambda^{(0)} d^{2N}\lambda^{(1)} \prod_{i=1}^{N} \frac{\mathrm{sh}^2\, 2\lambda_i^{(0)}}{\mathrm{ch}^{2n^{(0)}}\lambda_i^{(0)}} \prod_{I=1}^{2N} \frac{1}{\mathrm{ch}^{n^{(1)}}\lambda_I^{(1)}} \\
\times \frac{\prod_{i<j} \mathrm{sh}^2\left(\lambda_i^{(0)} - \lambda_j^{(0)}\right) \mathrm{sh}^2\left(\lambda_i^{(0)} + \lambda_j^{(0)}\right) \prod_{I<J} \mathrm{sh}^2\left(\lambda_I^{(1)} - \lambda_J^{(1)}\right)}{\prod_{i,J} \mathrm{ch}\left(\lambda_i^{(0)} - \lambda_J^{(1)}\right) \mathrm{ch}\left(\lambda_i^{(0)} + \lambda_J^{(1)}\right) \prod_{I<J} \mathrm{ch}\left(\lambda_I^{(1)} + \lambda_J^{(1)}\right)} \,.
\end{aligned} \tag{2.85}
$$

A first step is to write the contribution of the Sp node in terms of $2N$ eigenvalues satisfying

$$\lambda^{(0)}_{N+i} = -\lambda^{(0)}_i \,. \tag{2.86}$$

The interaction between the $Sp(2N)$ and $U(2N)$ nodes combine to a single Cauchy determinant (2.58)

$$\begin{aligned}
&\frac{\prod_{i<j}\operatorname{sh}^2\left(\lambda^{(0)}_i-\lambda^{(0)}_j\right)\operatorname{sh}^2\left(\lambda^{(0)}_i+\lambda^{(0)}_j\right)\prod_{i=1}^N\operatorname{sh}2\lambda^{(0)}_i\prod_{I<J}\operatorname{sh}\left(\lambda^{(1)}_I-\lambda^{(1)}_J\right)}{\prod_{i,J}\operatorname{ch}\left(\lambda^{(0)}_i-\lambda^{(1)}_J\right)\operatorname{ch}\left(\lambda^{(0)}_i+\lambda^{(1)}_J\right)}\\
&=\frac{\prod_{I<J}\operatorname{sh}\left(\lambda^{(0)}_I-\lambda^{(0)}_J\right)\operatorname{sh}\left(\lambda^{(1)}_I-\lambda^{(1)}_J\right)}{\prod_{I,J}\operatorname{ch}\left(\lambda^{(0)}_I-\lambda^{(1)}_J\right)}=\sum_{\tau^{(0)}\in S_{2N}}(-1)^{\tau^{(0)}}\prod_{I=1}^{2N}\frac{1}{\operatorname{ch}\left(\lambda^{(0)}_I-\lambda^{(1)}_{\tau^{(0)}(I)}\right)}\,.
\end{aligned}\tag{2.87}$$

The remaining terms involving the eigenvalues of the $U(2N)$ node can be interpreted as a Pfaffian, rather than a determinant. We can use the identity [74, 75]

$$\prod_{I<J\leqslant 2N}\frac{x_I-x_J}{1+x_Ix_J}=\operatorname{Pf}\left(\frac{x_I-x_J}{1+x_Ix_J}\right)=\frac{1}{2^NN!}\sum_{\tau\in S_{2N}}(-1)^\tau\prod_{i=1}^N\frac{x_{\tau(i)}-x_{\tau R(i)}}{1+x_{\tau(i)}x_{\tau R(i)}}\,, \tag{2.88}$$

where R is again the permutation $R(i)=N+i$ modulo $2N$. Plugging $x=e^{2\pi\lambda^{(1)}}$, we obtain

$$\prod_{I<J}\frac{\operatorname{sh}\left(\lambda^{(1)}_I-\lambda^{(1)}_J\right)}{\operatorname{ch}\left(\lambda^{(1)}_I+\lambda^{(1)}_J\right)}=\frac{1}{2^NN!}\sum_{\tau\in S_{2N}}(-1)^\tau\prod_{i=1}^N\frac{\operatorname{sh}\left(\lambda^{(1)}_{\tau(i)}-\lambda^{(1)}_{\tau R(i)}\right)}{\operatorname{ch}\left(\lambda^{(1)}_{\tau(i)}+\lambda^{(1)}_{\tau R(i)}\right)}\,. \tag{2.89}$$

As before, we can remove one of the permutations coming from (2.87) and (2.89) by a relabelling of eigenvalues, picking up an overall factor of $(2N)!$. This gives

$$\begin{aligned}
Z(N)=\frac{1}{2^{2N}N!^2}&\sum_{\tau\in S_{2N}}(-1)^\tau\int d^N\lambda^{(0)}d^{2N}\lambda^{(1)}\prod_{i=1}^N\frac{\operatorname{sh}2\lambda^{(0)}_i}{\operatorname{ch}^{2n^{(0)}}\lambda^{(0)}_i}\prod_{I=1}^{2N}\frac{1}{\operatorname{ch}^{n^{(1)}}\lambda^{(1)}_I}\\
&\prod_{i=1}^N\frac{1}{\operatorname{ch}\left(\lambda^{(0)}_{\tau(i)}-\lambda^{(1)}_{\tau(i)}\right)}\frac{\operatorname{sh}\left(\lambda^{(1)}_{\tau(i)}-\lambda^{(1)}_{\tau R(i)}\right)}{\operatorname{ch}\left(\lambda^{(1)}_{\tau(i)}+\lambda^{(1)}_{\tau R(i)}\right)}\frac{1}{\operatorname{ch}\left(\lambda^{(1)}_{\tau R(i)}-\lambda^{(0)}_{\tau R(i)}\right)}\,.
\end{aligned}\tag{2.90}$$

Replacing for convenience $\tau\to\tau^{-1}$, we can again rewrite the expression as a product over the set $\mathcal{K}(\tau)$, consisting of N indices closed under the permutation $R\tau^{-1}R\tau$

$$\begin{aligned}
Z(N)=\frac{1}{2^{2N}N!^2}&\sum_{\tau\in S_{2N}}(-1)^\tau\int d^N\lambda^{(0)}d^{2N}\lambda^{(1)}\prod_{k\in\mathcal{K}(\tau)}\frac{(-1)^{s(k)}\operatorname{sh}2\lambda^{(0)}_k}{\operatorname{ch}^{2n^{(0)}}\lambda^{(0)}_k}\frac{1}{\operatorname{ch}^{n^{(1)}}\lambda^{(1)}_k\operatorname{ch}^{n^{(1)}}\lambda^{(1)}_{R(k)}}\\
&\frac{1}{\operatorname{ch}\left(\lambda^{(0)}_k-\lambda^{(1)}_k\right)}\frac{(-1)^{s(\tau(k))}\operatorname{sh}\left(\lambda^{(1)}_k-\lambda^{(1)}_{\tau^{-1}R\tau(k)}\right)}{\operatorname{ch}\left(\lambda^{(1)}_k+\lambda^{(1)}_{\tau^{-1}R\tau(k)}\right)}\frac{1}{\operatorname{ch}\left(\lambda^{(1)}_{\tau^{-1}R\tau(k)}+\lambda^{(0)}_{R\tau^{-1}R\tau(k)}\right)}\,.
\end{aligned}\tag{2.91}$$

The $(-1)^{s(\tau(k))}$ signs comes from re-expressing $\operatorname{sh}\left(\lambda^{(1)}_{\tau(i)}-\lambda^{(1)}_{\tau R(i)}\right)$ in terms of k, while

(2.86) is responsible for the $(-1)^{s(k)}$ signs as well as allowing the replacement in the last denominator

$$\lambda^{(0)}_{\tau^{-1}R\tau(k)} = -\lambda^{(0)}_{R\tau^{-1}R\tau(k)}\,. \tag{2.92}$$

As in the case of the $\hat{D}$-quivers (2.77), we obtain a density operator between two $\lambda^{(0)}$ eigenvalues related by the permutation $R\tau^{-1}R\tau$

$$\rho(\lambda,\lambda') = \int d\lambda_1 d\lambda_2\, \frac{\operatorname{sh} 2\lambda}{\operatorname{ch}^{2n^{(0)}}\lambda}\frac{1}{\operatorname{ch}(\lambda-\lambda_1)}\frac{1}{\operatorname{ch}^{n^{(1)}}\lambda_1}\frac{\operatorname{sh}(\lambda_1-\lambda_2)}{\operatorname{ch}(\lambda_1+\lambda_2)}\frac{1}{\operatorname{ch}^{n^{(1)}}\lambda_2}\frac{1}{\operatorname{ch}(\lambda_2+\lambda')}\,, \tag{2.93}$$

in terms of which the partition function is given exactly as in (2.77). Expanding $\operatorname{sh}(\lambda_1-\lambda_2)$ and reversing the sign of λ_2 allows us again to represent the operator in terms of canonical position and momentum operators

$$\begin{aligned}\rho &= \frac{1}{2}\frac{\operatorname{sh} 2q}{\operatorname{ch}^{2n^{(0)}}q}\frac{1}{\operatorname{ch} p}\frac{1}{\operatorname{ch}^{n^{(1)}}q}\left(\operatorname{sh} q\frac{1}{\operatorname{ch} p}\operatorname{ch} q + \operatorname{ch} q\frac{1}{\operatorname{ch} p}\operatorname{sh} q\right)\frac{1}{\operatorname{ch}^{n^{(1)}}q}\frac{1}{\operatorname{ch} p}\\ &= \frac{1}{2}\frac{\operatorname{sh} 2q}{\operatorname{ch}^{2n^{(0)}}q}\frac{1}{\operatorname{ch} p}\frac{1}{\operatorname{ch}^{n^{(1)}}q}\left(e^{\pi q}\frac{1}{\operatorname{ch} p}e^{\pi q} + e^{-\pi q}\frac{1}{\operatorname{ch} p}e^{-\pi q}\right)\frac{1}{\operatorname{ch}^{n^{(1)}}q}\frac{1}{\operatorname{ch} p}\,.\end{aligned} \tag{2.94}$$

The same arguments as for the $\hat{D}$-quiver allow us to express $Z(N)$ as the partition function of N non-interacting fermions on a half line with Neumann (2.83) or Dirichlet (2.84) boundary conditions.

2.4.4 Generalisation to longer quivers

It is straightforward to generalise the analysis to quivers with an arbitrary number of $U(2N)$ nodes and arbitrary number of fundamental hypermultiplets on each node. Just as for the $\hat{A}$-quiver theories, the matrix model contributions from hypermultiplets transforming in the bifundamental representation of pairs of $U(2N)$ gauge nodes combine with the vector multiplet contributions to form one Cauchy determinant between each pair of adjacent nodes. This translates into $\operatorname{ch}^{-1} p$ terms in the density operator. We represent the contribution from fundamental hypermultiplets and the FI and CS terms of the a^{th} node again by $F^{(a)}(q)$ (2.57). This leads to a piece in the density operator of the form

$$\frac{1}{\operatorname{ch} p}F^{(1)}(q)\frac{1}{\operatorname{ch} p}F^{(2)}(q)\frac{1}{\operatorname{ch} p}F^{(3)}(q)\frac{1}{\operatorname{ch} p}\cdots \tag{2.95}$$

In all of our examples the density operator combines kernels going back and forth along the quiver. The contribution from the ends of the quivers ($U(N)$ nodes for $\hat{D}$-quivers and $Sp(2N)$/antisymmetric hypermultiplet for linear quivers) are the same as in the previous sections.

For the $\hat{D}$-quivers the contribution from going back along the quiver as in (2.72) gives

$$\cdots \frac{1}{\operatorname{ch} p} F^{(3)}(q) \frac{1}{\operatorname{ch} p} F^{(2)}(q) \frac{1}{\operatorname{ch} p} F^{(1)}(q) \frac{1}{\operatorname{ch} p} \,. \tag{2.96}$$

For the linear quivers, the antisymmetric hypermultiplet or Sp node introduce a minus sign, like the replacement $\lambda_2 \to -\lambda_2$ that gave (2.94) from (2.93). Therefore the second part of the density operator includes

$$\cdots \frac{1}{\operatorname{ch} p} F^{(3)}(-q) \frac{1}{\operatorname{ch} p} F^{(2)}(-q) \frac{1}{\operatorname{ch} p} F^{(1)}(-q) \frac{1}{\operatorname{ch} p} \,. \tag{2.97}$$

Generalised $\hat{D}$-quivers

Let us consider a $\hat{D}_{L+2}$ quiver with arbitrary number of gauge nodes and fundamental hypermultiplets on each node,[7] as shown in figure 2.2. We label the $U(2N)$ nodes by $1, \cdots, L-1$, and as in figure 2.2, we distinguish parameters from pairs of terminal $U(N)$ nodes by primes: $F^{(0)}$, $F'^{(0)}$,$F^{(L)}$, $F'^{(L)}$. We note that all bifundamental hypermultiplet masses can be set to zero by shifting eigenvalues.[8] The above rules lead to the density operator

$$\begin{aligned}\rho =& \frac{1}{4} \frac{1}{\operatorname{ch} p} \left(F^{(0)}(q) \frac{\operatorname{sh} p}{\operatorname{ch} p} F'^{(0)}(q) + F'^{(0)}(q) \frac{\operatorname{sh} p}{\operatorname{ch} p} F^{(0)}(q) \right) \frac{1}{\operatorname{ch} p} \left(\prod_{a=1}^{L-1} F^{(a)}(q) \frac{1}{\operatorname{ch} p} \right) \\ & \left(F^{(L)}(q) \frac{\operatorname{sh} p}{\operatorname{ch} p} F'^{(L)}(q) + F'^{(L)}(q) \frac{\operatorname{sh} p}{\operatorname{ch} p} F^{(L)}(q) \right) \frac{1}{\operatorname{ch} p} \left(\prod_{a=1}^{L-1} F^{(L-a)}(q) \frac{1}{\operatorname{ch} p} \right) .\end{aligned} \tag{2.98}$$

We can easily recover (2.79) by setting $L = 2$, $F^{(2)}(q) = \operatorname{ch}^{-n} q$ and all other $F^{(a)} = F'^{(a)} = 1$.

Generalised linear quivers

We can proceed in a similar fashion to write down the density operators for longer linear quivers, where each end of the $U(2N)$ linear chain has either an $Sp(2N)$ node or an antisymmetric hypermultiplet and any number of fundamental hypermultiplets on all the nodes.[9] Again we note that the masses for all bifundamental hypermultiplets between $U(2N)$ nodes can be set to zero by shifts of the eigenvalues.

[7] Note that in order for the matrix model to be convergent, such a theory must have at least one fundamental hypermultiplet. With this condition violated the formal manipulations still go through, but the divergence of the matrix model translates to a density operator which is not of trace class.

[8] Shifting eigenvalues to remove masses can introduce an overall phase in the matrix model. Such phases are unphysical, in the sense that they arise from background (mixed) Chern-Simons terms that can be added to the regularisation scheme when computing the partition function of the 3d theories [76, 77].

[9] While the formal manipulations again go through in all cases (see also footnote 7), convergence of the matrix model requires a total of at least three fundamental hypermultipets, with at least one coupling to the terminating nodes at each end of the quiver.

We cannot always do the same for the mass of antisymmetric hypermultiplets, or for those charged under $Sp(2N)$, so we keep these masses as well as the fundamental hypermultiplet masses. We consider a quiver with L $U(2N)$ nodes and again package the FI, CS and fundamental hypermultiplet contributions into $F^{(a)}$. Two instances (out of three) of such general linear quivers are pictured in figure 2.4. The density operator is given by

$$\rho = B^{(0)}(p,q)\left(\prod_{a=1}^{L-1} F^{(a)}(q)\frac{1}{\operatorname{ch} p}\right)F^{(L)}(q)B^{(L+1)}(p,q)\left(\prod_{a=1}^{L-1} F^{(L+1-a)}(-q)\frac{1}{\operatorname{ch} p}\right)F^{(1)}(-q)\,, \tag{2.99}$$

where the functions $B^{(a)}(p,q)$ account for whether the ends of the quiver terminate with an $Sp(2N)$ node or antisymmetric hypermultiplet.

If the quiver terminates with an $Sp(2N)$ node we have

$$B^{(a)}_{Sp}(p,q) = \frac{e^{2\pi i m^{(a)} p}}{\operatorname{ch} p}\operatorname{sh}(2q)\widetilde{F}^{(a)}(q)\frac{e^{2\pi i m^{(a)} p}}{\operatorname{ch} p}\,, \tag{2.100}$$

where $m^{(a)}$ is the bifundamental mass and $\widetilde{F}^{(a)}(q)$ captures the contributions from the CS term with level $k^{(a)}$ and $n^{(a)}$ fundamental hypermultiplets of $Sp(2N)$ with masses $\mu^{(a)}_\alpha$

$$\widetilde{F}^{(a)}(q) = \frac{e^{2\pi i k^{(a)} q^2}}{\prod_{\alpha=1}^{n^{(a)}}\operatorname{ch}(q-\mu^{(a)}_\alpha)\operatorname{ch}(q+\mu^{(a)}_\alpha)}\,. \tag{2.101}$$

If it terminates with an antisymmetric hypermultiplet we have

$$B^{(a)}_{A}(p,q) = \frac{1}{2}\left(e^{\pi q}\frac{e^{2\pi i M^{(a)} p}}{\operatorname{ch} p}e^{\pi q} + e^{-\pi q}\frac{e^{2\pi i M^{(a)} p}}{\operatorname{ch} p}e^{-\pi q}\right)\,, \tag{2.102}$$

where $M^{(a)}$ is the antisymmetric mass.

Note that the expression (2.99) assumes that there is at least one $U(2N)$ node. There are two relevant cases without $U(2N)$ nodes: For the single node $Sp(2N)$ theory with an antisymmetric hypermultiplet the density operator is given by[10]

$$\rho = \frac{1}{2}\operatorname{sh}(2q)\widetilde{F}^{(0)}(q)\left(\operatorname{sh} q\frac{e^{2\pi i M p}}{\operatorname{ch} p}\operatorname{ch} q + \operatorname{ch} q\frac{e^{2\pi i M p}}{\operatorname{ch} p}\operatorname{sh} q\right)\,. \tag{2.103}$$

For the $Sp(2N)\times Sp(2N)$ theory the density operator is given by

$$\rho = \frac{e^{2\pi i m^{(0)} p}}{\operatorname{ch} p}\operatorname{sh}(2q)\widetilde{F}^{(0)}(q)\frac{e^{2\pi i m^{(0)} p}}{\operatorname{ch} p}\operatorname{sh}(2q)\widetilde{F}^{(1)}(q)\,. \tag{2.104}$$

[10]This theory was previously studied from the Fermi gas perspective in [36], where a rather different density operator was obtained. We compare the different formalisms in appendix C.

2.5 Mirror symmetry as a canonical transformation

As discussed in the introduction, 3d theories with at least $\mathcal{N} = 4$ supersymmetry exhibit mirror symmetry [42], which relates theories that flow to the same infra red fixed point.

Since the partition function is independent of the running couplings, a useful test of mirror symmetry is that mirror dual theories must have equal partition functions. Such tests were successfully carried out at the level of matrix models for mirror pairs involving both $\hat{A}$ and $\hat{D}$ quivers shortly after the advent of localisation in [74,78–81], where matrix models of dual pairs were matched using integral identities.

The advantage of working in the Fermi gas formalism is that complicated matrix models are encoded in single particle density operators, and so we can hope that mirror dualities are realised in a simple way at the level of the density operator. Indeed, we find that density operators of mirror dual theories are related by very simple canonical transformations, and that these canonical transformations allow one to very easily generate pairs of mirror theories. In particular this approach allows for an efficient determination of the mirror maps - the linear relations between FI parameters on one side of the duality and mass parameters on the other. We show how this works for a number of examples and comment also on more general cases.

2.5.1 $\hat{A}_1$ quiver with two fundamentals

In this section we examine a 2 node $\hat{A}$ quiver with a single fundamental hypermultiplet for each node, arbitrary bifundamental mass and FI parameters and vanishing CS levels (see top quiver in figure 2.6), where the density operator (2.66) becomes

$$\rho = \frac{e^{2\pi i\zeta^{(1)}q}}{\operatorname{ch} q}\frac{e^{2\pi i m^{(1)}p}}{\operatorname{ch} p}\frac{e^{2\pi i\zeta^{(2)}q}}{\operatorname{ch} q}\frac{e^{2\pi i m^{(2)}p}}{\operatorname{ch} p}\,. \tag{2.105}$$

This theory has two mirror theories, related in the IIB brane construction by $SL(2,\mathbb{Z})$ transformations.

S transformation

The first of these mirror theories is one with identical matter content but with mass and FI parameters exchanged [44], as illustrated by the bottom right quiver in figure 2.6.

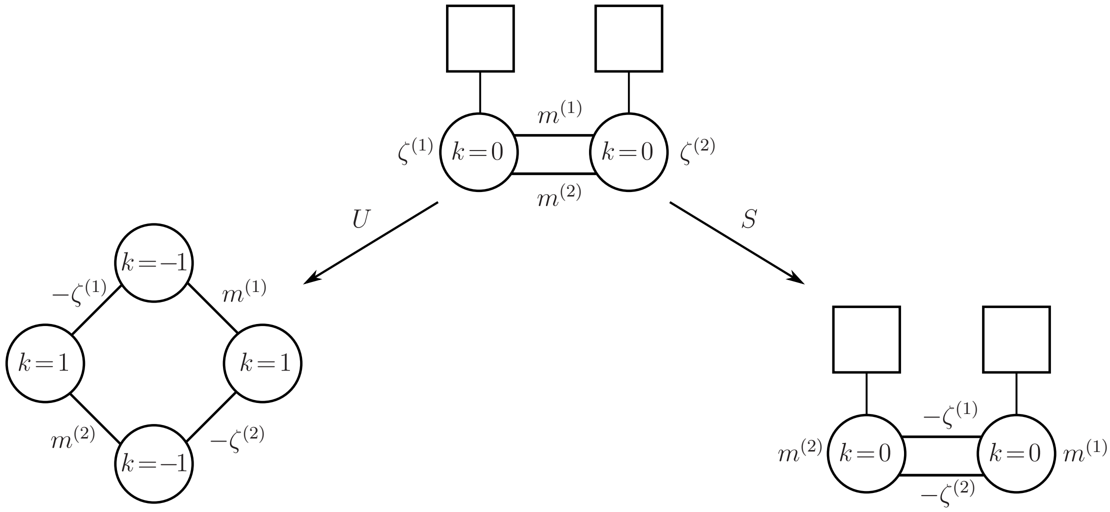

Figure 2.6: Quiver diagrams of the two node theory we discuss in the text and its two mirror duals. Each circle represents a $U(N)$ vector multiplet, labelled inside by the CS level k and outside by the FI parameter. Edges represent hypermultiplets. Those connecting two circles are bifundamental fields with the mass indicated next to them. The boxes represent $U(1)$ flavour symmetries of fundamental hypers, which in our examples are massless.

At the level of the density operator this is captured by the replacement[11]

$$p \to q\,, \qquad q \to -p\,. \tag{2.106}$$

Applying this to (2.105) gives

$$\rho^{(S)} = \frac{e^{-2\pi i\zeta^{(1)}p}}{\operatorname{ch} p}\frac{e^{2\pi i m^{(1)}q}}{\operatorname{ch} q}\frac{e^{-2\pi i\zeta^{(2)}p}}{\operatorname{ch} p}\frac{e^{2\pi i m^{(2)}q}}{\operatorname{ch} q} \sim \frac{e^{2\pi i m^{(2)}q}}{\operatorname{ch} q}\frac{e^{-2\pi i\zeta^{(1)}p}}{\operatorname{ch} p}\frac{e^{2\pi i m^{(1)}q}}{\operatorname{ch} q}\frac{e^{-2\pi i\zeta^{(2)}p}}{\operatorname{ch} p} \tag{2.107}$$

where the last relation represents equivalence under conjugating by $\frac{e^{2\pi i m^{(2)}q}}{\operatorname{ch} q}$. This density operator is indeed the same as (2.105) under the replacements

$$\zeta^{(1)} \to m^{(2)}, \quad \zeta^{(2)} \to m^{(1)}, \quad m^{(1)} \to -\zeta^{(1)}, \quad m^{(2)} \to -\zeta^{(2)} \tag{2.108}$$

[11] To see that the partition function is really invariant under these replacements, note that they can be equivalently thought of as a similarity transformation of the density operator $\rho \to V^{-1}\rho V$. Taking $V = e^{i\pi q^2}e^{i\pi p^2}e^{i\pi q^2}$ and using the relations (2.63) to conjugate the exponential factors through ρ brings about the replacements (2.106). Since the spectrum of ρ is unaffected by similarity transformations, this guarantees the invariance of the partition function under the replacements (2.106).

U transformation

To get the second mirror theory we apply to (2.105) the replacement[12]

$$p \to p + q \,, \qquad q \to -p \,. \tag{2.109}$$

The result is

$$\begin{aligned}
\rho^{(U)} &= \frac{e^{-2\pi i\zeta^{(1)}p}}{\operatorname{ch} p}\frac{e^{2\pi i m^{(1)}(p+q)}}{\operatorname{ch}(p+q)}\frac{e^{-2\pi i\zeta^{(2)}p}}{\operatorname{ch} p}\frac{e^{2\pi i m^{(2)}(p+q)}}{\operatorname{ch}(p+q)} \\
&= \frac{e^{-2\pi i\zeta^{(1)}p}}{\operatorname{ch} p}e^{-i\pi q^2}\frac{e^{2\pi i m^{(1)}p}}{\operatorname{ch} p}e^{i\pi q^2}\frac{e^{-2\pi i\zeta^{(2)}p}}{\operatorname{ch} p}e^{-i\pi q^2}\frac{e^{2\pi i m^{(2)}p}}{\operatorname{ch} p}e^{i\pi q^2} \,.
\end{aligned} \tag{2.110}$$

In the second line we have made use of the identity (2.63)

$$e^{-\pi i q^2} f(p) e^{\pi i q^2} = f(p+q) \,. \tag{2.111}$$

To read off the corresponding quiver theory from (2.110), each $e^{i\pi(kq^2+2\zeta q)}$ term can be associated to a $U(N)$ node with CS level k and FI parameter ζ, while each $\frac{e^{2\pi i m p}}{\operatorname{ch} p}$ comes from a bifundamental hypermultiplet with mass m. The transformed density operator corresponds therefore to a circular quiver with four nodes that have alternating Chern-Simons levels $k = \pm 1$ and vanishing FI parameters. The bifundamental multiplets connecting adjacent nodes have masses $\{-\zeta^{(1)}, m^{(1)}, -\zeta^{(2)}, m^{(2)}\}$, as in the bottom left diagram of figure 2.6.

$SL(2,\mathbb{Z})$

It is easy to see that the transformations we used in the previous sections close onto $SL(2,\mathbb{Z})$. Indeed, defining $T = SU$ we find the defining relations

$$S^2 = -I \,, \qquad (ST)^3 = I \,. \tag{2.112}$$

More general $SL(2,\mathbb{Z})$ transformations will give density operators with terms of the form

$$\frac{1}{\operatorname{ch}(ap + bq)} \,. \tag{2.113}$$

The cases with $a = 0$ and $b = 1$ or $a = \pm 1$ and $b \in \mathbb{Z}$ have a natural interpretation as a contribution of a fundamental field, or as we have seen in (2.111), from conjugating the usual $1/\cosh \pi p$ by CS terms. But these manipulations cannot undo expressions one finds from a general $SL(2,\mathbb{Z})$ transformation of the density operator. In these more general cases, the transformed density operator can still be associated to a

[12]Again this replacement is equivalent to a similarity transformation $\rho \to V^{-1}\rho V$, in the case with $V = e^{i\pi p^2} e^{i\pi q^2}$ (see also footnote 11)

matrix model, but it cannot be derived from any known 3d Lagrangian.

This is also manifested in the IIB brane realisation, where any $SL(2,\mathbb{Z})$ transformation will lead to some configuration of (p,q) branes. Most of those do not have a known Lagrangian description [82], but one could associate to them a matrix model [53, 83], which would indeed lead to the transformed density operator.

2.5.2 Mirror symmetry for generic $\mathcal{N}=4$ $\hat{A}$ quivers

The manifestation of mirror symmetry as a canonical transformation naturally generalises to the entire family of $\mathcal{N}=4$ $\hat{A}$ quivers with an arbitrary number of nodes. To see this, let us apply the S and U transformations of the previous sections to the density operator (2.66), which in the absence of CS terms and fundamental masses becomes

$$\rho = \prod_{a=1}^{L} \frac{e^{2\pi i \zeta^{(a)} q}}{\mathrm{ch}^{n^{(a)}} q} \frac{e^{2\pi i m_a p}}{\mathrm{ch}\, p}\,, \tag{2.114}$$

Applying the S transformation we get

$$\rho^{(S)} = \prod_{a=1}^{L} \frac{e^{2\pi i \zeta^{(a)} p}}{\mathrm{ch}^{n^{(a)}} p} \frac{e^{2\pi i m^{(a)} q}}{\mathrm{ch}\, q}\,. \tag{2.115}$$

This density operator is that of an $\hat{A}$ quiver theory with $\sum_{a=1}^{L} n^{(a)}$ nodes and L fundamental matter fields. The fundamentals are attached to nodes which have FI parameters $m^{(a)}$, and are separated by $n^{(a-1)}$ other nodes. The masses of the bifundamentals connecting them add up to $-\zeta^{(a)}$.[13]

Applying the U transformation we get

$$\rho^{(U)} = \prod_{a=1}^{L} \frac{e^{-2\pi i \zeta^{(a)} p}}{\mathrm{ch}^{n^{(a)}} p} \frac{e^{2\pi i m^{(a)} (p+q)}}{\mathrm{ch}(p+q)} = \prod_{a=1}^{L} \frac{e^{-2\pi i \zeta^{(a)} p}}{\mathrm{ch}^{n^{(a)}} p} e^{-\pi i q^2} \frac{e^{2\pi i m^{(a)} p}}{\mathrm{ch}\, p} e^{\pi i q^2}\,. \tag{2.116}$$

The mirror theory can be readily read off from this density operator as a circular quiver theory with $\sum_{a=1}^{n} N_a + n$ nodes and no fundamental matter. Each node has Chern-Simons level $k = +1, -1$ or 0. Further details concerning the mass parameters and value of the Chern-Simons level at each node can be read off in much the same way as for the previous example.

A final generalisation (that preserves $\mathcal{N}=4$ supersymmetry) is to turn on masses for the fundamental fields. This corresponds to replacing each of the $\mathrm{ch}^{n^{(a)}} q$ in

[13] At the level of the matrix model, this additional freedom to choose mass parameters in the mirror theory simply amounts to the freedom to make constant shifts in the integration variables.

(2.114) with a product of $n^{(a)}$ terms with masses μ_α

$$\begin{aligned} e^{2\pi i\zeta^{(a)}q}\prod_{\alpha=1}^{n^{(a)}}\frac{1}{\operatorname{ch}(q+\mu_\alpha)} &= e^{2\pi i\zeta^{(a)}q}\prod_{\alpha=1}^{n^{(a)}}\left(e^{2\pi i\mu_\alpha p}\frac{1}{\operatorname{ch}q}e^{-2\pi i\mu_\alpha p}\right) \\ &= e^{-2\pi i\zeta^{(a)}\mu_1}e^{2\pi i\mu_1 p}\frac{e^{2\pi i\zeta^{(a)}q}}{\operatorname{ch}q}e^{-2\pi i\mu_1 p}\prod_{\alpha=2}^{n^{(a)}}\left(e^{2\pi i\mu_\alpha p}\frac{1}{\operatorname{ch}q}e^{-2\pi i\mu_\alpha p}\right)\,, \end{aligned} \tag{2.117}$$

where in the second line we chose to associate the FI term to the first fundamental field, picking up an overall phase that can be neglected (see footnote 8).

Once we apply S or U transformations to (2.117) it becomes clear that these mass terms become additional FI parameters, as is expected.

2.5.3 Mirror symmetry for $\hat{D}$ quivers

Unlike $\hat{A}$ quivers, the mirror duals of $\hat{D}$ quivers are not other $\hat{D}$ quivers, but rather belong to the class of $SP-U$ linear quivers discussed in section 2.4.3 [43]. Also, unlike $\hat{A}$ quivers, each $\hat{D}$ quiver in general has only a single mirror dual, corresponding to an S-duality transformation in the IIB construction. There are some exceptions to this, for instance a $\hat{D}_4$ quiver can have multiple mirror duals, and we give an example of this in the following section. We again find that mirror symmetry can be identified with a canonical transformation

$$p \to q, \qquad q \to -p\,, \tag{2.118}$$

acting on the density operator.

2.5.4 $\hat{D}_4$-quiver with two fundamentals

The first example we consider is the $\hat{D}_4$-quiver with a fundamental hypermultiplet on two of the terminal $U(N)$ nodes. This example is somewhat special as there are three inequivalent ways of pairing up the terminal $U(N)$ nodes, which leads to different density operators. The canonical transformations of these three descriptions are related to three different mirror theories. The existence of several mirror dual theories was already noted in section 4.4.3 of [81].

The first possibility is to pair the two terminal nodes without fundamental matter and the two with fundamental matter, as shown in figure 2.7. Within each pairing we distinguish the FI parameters of the two $U(N)$ nodes by giving one of them a prime. We do not turn on masses for the two $U(N)$ fundamental hypermultiplets.[14]

[14] One of these mass parameters can be removed by shifts of the matrix model eigenvalues. The other is mapped under mirror symmetry to a "hidden" FI parameter [46, 84], which does not have a clear interpretation in the mirror gauge theory.

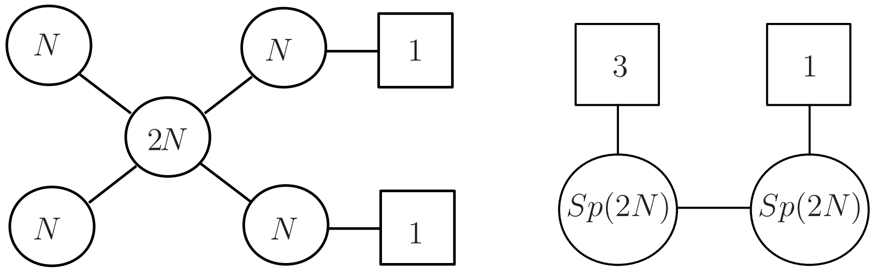

Figure 2.7: $\hat{D}_4$-quiver and its mirror dual theory with $Sp(2N)\times Sp(2N)$ gauge group.

The density operator of the $\hat{D}_4$ theory can be read from (2.98)

$$\begin{aligned}\rho = &\frac{1}{4}\frac{1}{\operatorname{ch} p}\left(e^{2\pi i\zeta^{(0)}q}\frac{\operatorname{sh} p}{\operatorname{ch} p}e^{2\pi i\zeta'^{(0)}q} + e^{2\pi i\zeta'^{(0)}q}\frac{\operatorname{sh} p}{\operatorname{ch} p}e^{2\pi i\zeta^{(0)}q}\right)\frac{1}{\operatorname{ch} p}e^{2\pi i\zeta^{(1)}q}\\ &\times\frac{1}{\operatorname{ch} p}\frac{1}{\operatorname{ch} q}\left(e^{2\pi i\zeta^{(2)}q}\frac{\operatorname{sh} p}{\operatorname{ch} p}e^{2\pi i\zeta'^{(2)}q} + e^{2\pi i\zeta'^{(2)}q}\frac{\operatorname{sh} p}{\operatorname{ch} p}e^{2\pi i\zeta^{(2)}q}\right)\frac{1}{\operatorname{ch} q}\frac{1}{\operatorname{ch} p}e^{2\pi i\zeta^{(1)}q}\,.\end{aligned} \tag{2.119}$$

To map this to the density operator of the mirror dual theory, we first use the relation

$$e^{2\pi i\zeta q}f(p)e^{-2\pi i\zeta q} = f(p-\zeta)\,, \tag{2.120}$$

to simplify the terms in parenthesis

$$\begin{aligned}&e^{\pi i(\zeta-\zeta')q}\frac{\operatorname{sh} p}{\operatorname{ch} p}e^{-\pi i(\zeta-\zeta')q} + e^{-\pi i(\zeta-\zeta')q}\frac{\operatorname{sh} p}{\operatorname{ch} p}e^{\pi i(\zeta-\zeta')q}\\ &= \frac{\operatorname{sh}(p+\frac{1}{2}\zeta'-\frac{1}{2}\zeta)}{\operatorname{ch}(p+\frac{1}{2}\zeta'-\frac{1}{2}\zeta)} + \frac{\operatorname{sh}(p-\frac{1}{2}\zeta'+\frac{1}{2}\zeta)}{\operatorname{ch}(p-\frac{1}{2}\zeta'+\frac{1}{2}\zeta)} = \frac{2\operatorname{sh} 2p}{\operatorname{ch}(p+\frac{1}{2}\zeta-\frac{1}{2}\zeta')\operatorname{ch}(p-\frac{1}{2}\zeta+\frac{1}{2}\zeta')}\,.\end{aligned} \tag{2.121}$$

By further commuting exponential factors we get

$$\begin{aligned}\rho = &e^{\pi i(\zeta^{(0)}+\zeta'^{(0)})q}\frac{1}{\operatorname{ch}\left(p+\tilde{\mu}_1^{(1)}\right)}\frac{\operatorname{sh} 2p}{\operatorname{ch}\left(p+\tilde{\mu}_2^{(1)}\right)\operatorname{ch}\left(p-\tilde{\mu}_2^{(1)}\right)}\frac{1}{\operatorname{ch}\left(p-\tilde{\mu}_1^{(1)}\right)}\frac{1}{\operatorname{ch}\left(p+\tilde{\mu}_3^{(1)}\right)}\\ &\times\frac{e^{-\pi i\tilde{m}q}}{\operatorname{ch} q}\frac{\operatorname{sh} 2p}{\operatorname{ch}\left(p+\tilde{\mu}^{(0)}\right)\operatorname{ch}\left(p-\tilde{\mu}^{(0)}\right)}\frac{e^{-\pi i\tilde{m}q}}{\operatorname{ch} q}\frac{1}{\operatorname{ch}\left(p-\tilde{\mu}_3^{(1)}\right)}e^{-\pi i(\zeta^{(0)}+\zeta'^{(0)})q}\,,\end{aligned} \tag{2.122}$$

with

$$\begin{aligned}\tilde{\mu}_1^{(1)} &= \tfrac{1}{2}(\zeta^{(0)}+\zeta'^{(0)})\,,\quad \tilde{\mu}_2^{(1)} = \tfrac{1}{2}(\zeta^{(0)}-\zeta'^{(0)})\,,\quad \tilde{\mu}_3^{(1)} = -\zeta^{(1)}-\tfrac{1}{2}(\zeta^{(0)}+\zeta'^{(0)})\,,\\ \tilde{\mu}^{(0)} &= \tfrac{1}{2}(\zeta^{(2)}-\zeta'^{(2)})\,,\\ \tilde{m} &= -\zeta^{(1)}-\tfrac{1}{2}(\zeta^{(0)}+\zeta'^{(0)}+\zeta^{(2)}+\zeta'^{(2)})\,.\end{aligned} \tag{2.123}$$

Now we can act on the density operator by canonical transformation (2.118). In addition we conjugate the operator to remove the exponential factors at the beginning and the end, which clearly does not alter the spectrum. This gives

$$\tilde{\rho} = \frac{e^{2\pi i \tilde{m} p}}{\operatorname{ch} p} \frac{\operatorname{sh} 2q}{\operatorname{ch}\left(q + \tilde{\mu}^{(0)}\right) \operatorname{ch}\left(q - \tilde{\mu}^{(0)}\right)} \frac{e^{2\pi i \tilde{m} p}}{\operatorname{ch} p} \frac{\operatorname{sh} 2q}{\prod_{\alpha=1}^{3} \operatorname{ch}\left(q + \tilde{\mu}_\alpha^{(1)}\right) \operatorname{ch}\left(q - \tilde{\mu}_\alpha^{(1)}\right)} . \tag{2.124}$$

As advertised we recover the density operator for a linear quiver with two $Sp(2N)$ nodes. One with a single fundamental hypermultiplet and the other with three (2.104). The relations between the FI and mass deformation parameters of the mirror dual theories are expressed in (2.123), where $\tilde{m}$ is the bifundamental hypermultiplet mass and $\tilde{\mu}^{(0)}, \tilde{\mu}_\alpha^{(1)}$ are the fundamental hypermultiplet masses of the dual theory. This mirror map generalises slightly the one found already in [74], allowing for $\zeta^{(0)} \neq \zeta'^{(0)}$ and $\zeta^{(2)} \neq \zeta'^{(2)}$, which translate to additional mass deformations in the dual linear quiver theory.

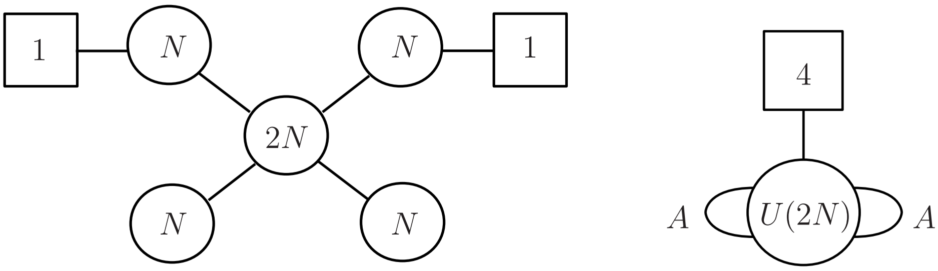

Figure 2.8: $\hat{D}_4$-quiver and its mirror dual theory with $U(2N)$ gauge group and antisymmetric hypermultiplets.

A second way of obtaining a density operator for the $\hat{D}_4$-quiver comes from pairing nodes with and without fundamental hypermultiplet, as in figure 2.8. After shifting eigenvalues to remove masses we are left with only one nonzero mass μ for one of the fundamental hypermultiplets. From (2.98) we can write down the density operator

$$\begin{aligned} \rho = & \frac{1}{4} \frac{1}{\operatorname{ch} p} \left(\frac{e^{2\pi i \zeta^{(0)} q}}{\operatorname{ch} q} \frac{\operatorname{sh} p}{\operatorname{ch} p} e^{2\pi i \zeta'^{(0)} q} + e^{2\pi i \zeta'^{(0)} q} \frac{\operatorname{sh} p}{\operatorname{ch} p} \frac{e^{2\pi i \zeta^{(0)} q}}{\operatorname{ch} q} \right) \frac{1}{\operatorname{ch} p} e^{2\pi i \zeta^{(1)} q} \\ & \times \frac{1}{\operatorname{ch} p} \left(\frac{e^{2\pi i \zeta^{(2)} q}}{\operatorname{ch}(q+\mu)} \frac{\operatorname{sh} p}{\operatorname{ch} p} e^{2\pi i \zeta'^{(2)} q} + e^{2\pi i \zeta'^{(2)} q} \frac{\operatorname{sh} p}{\operatorname{ch} p} \frac{e^{2\pi i \zeta^{(2)} q}}{\operatorname{ch}(q+\mu)} \right) \frac{1}{\operatorname{ch} p} e^{2\pi i \zeta^{(1)} q} . \end{aligned} \tag{2.125}$$

Once again we start by manipulating the expression of the density operator, using the relation (2.120)

$$\begin{aligned}\rho=&\frac{1}{4}e^{-i\pi\zeta^{(1)}q}\frac{1}{\mathrm{ch}(p+\frac{\zeta^{(1)}}{2})}\frac{1}{\mathrm{ch}(p-\frac{\zeta^{(1)}}{2})}\\&\times\left(\frac{e^{2\pi i(\zeta^{(0)}+\zeta'^{(0)}+\zeta^{(1)})q}}{\mathrm{ch}\,q}\frac{\mathrm{sh}(p+\zeta'^{(0)}+\frac{\zeta^{(1)}}{2})}{\mathrm{ch}(p+\zeta'^{(0)}+\frac{\zeta^{(1)}}{2})}+\frac{\mathrm{sh}(p-\zeta'^{(0)}-\frac{\zeta^{(1)}}{2})}{\mathrm{ch}(p-\zeta'^{(0)}-\frac{\zeta^{(1)}}{2})}\frac{e^{2\pi i(\zeta^{(0)}+\zeta'^{(0)}+\zeta^{(1)})q}}{\mathrm{ch}\,q}\right)\\&\times\frac{1}{\mathrm{ch}(p+\frac{\zeta^{(1)}}{2})}\frac{1}{\mathrm{ch}(p-\frac{\zeta^{(1)}}{2})}\\&\times\left(\frac{e^{2\pi i(\zeta^{(2)}+\zeta'^{(2)}+\zeta^{(1)})q}}{\mathrm{ch}(q+\mu)}\frac{\mathrm{sh}(p+\zeta'^{(2)}+\frac{\zeta^{(1)}}{2})}{\mathrm{ch}(p+\zeta'^{(2)}+\frac{\zeta^{(1)}}{2})}+\frac{\mathrm{sh}(p-\zeta'^{(2)}-\frac{\zeta^{(1)}}{2})}{\mathrm{ch}(p-\zeta'^{(2)}-\frac{\zeta^{(1)}}{2})}\frac{e^{2\pi i(\zeta^{(2)}+\zeta'^{(2)}+\zeta^{(1)})q}}{\mathrm{ch}(q+\mu)}\right)\\&\times e^{i\pi\zeta^{(1)}q}.\end{aligned}\tag{2.126}$$

We now use the identity

$$\begin{aligned}&\frac{e^{2\pi i\zeta q}}{\mathrm{ch}(q+\mu)}\frac{\mathrm{sh}\,p+\zeta'}{\mathrm{ch}\,p+\zeta'}+\frac{\mathrm{sh}\,p-\zeta'}{\mathrm{ch}\,p-\zeta'}\frac{e^{2\pi i\zeta q}}{\mathrm{ch}(q+\mu)}\\&\quad=e^{-2\pi i\zeta\mu}\frac{e^{2\pi i\mu p}}{\mathrm{ch}(p-\zeta')}\left(e^{\pi p}\frac{e^{2\pi i\zeta q}}{\mathrm{ch}\,q}e^{\pi p}-e^{-\pi p}\frac{e^{2\pi i\zeta q}}{\mathrm{ch}\,q}e^{-\pi p}\right)\frac{e^{-2\pi i\mu p}}{\mathrm{ch}(p+\zeta')}\,,\end{aligned}\tag{2.127}$$

to bring the density operator into the form

$$\begin{aligned}\rho=&\frac{1}{4}e^{-2\pi i(\zeta^{(1)}+\zeta^{(2)}+\zeta'^{(2)})\mu}e^{-\pi i\zeta^{(1)}q}e^{2\pi i\mu p}\\&\times\frac{e^{-2\pi i\mu p}}{\mathrm{ch}\left(p+\frac{1}{2}\zeta^{(1)}+\zeta'^{(2)}\right)}\frac{1}{\mathrm{ch}\left(p+\frac{1}{2}\zeta^{(1)}\right)}\frac{1}{\mathrm{ch}\left(p-\frac{1}{2}\zeta^{(1)}\right)}\frac{1}{\mathrm{ch}\left(p-\zeta'^{(0)}-\frac{1}{2}\zeta^{(1)}\right)}\\&\times\left(e^{\pi p}\frac{e^{2\pi i(\zeta^{(0)}+\zeta'^{(0)}+\zeta^{(1)})q}}{\mathrm{ch}\,q}e^{\pi p}+e^{-\pi p}\frac{e^{2\pi i(\zeta^{(0)}+\zeta'^{(0)}+\zeta^{(1)})q}}{\mathrm{ch}\,q}e^{-\pi p}\right)\\&\times\frac{e^{2\pi i\mu p}}{\mathrm{ch}\left(p+\zeta'^{(0)}+\frac{1}{2}\zeta^{(1)}\right)}\frac{1}{\mathrm{ch}\left(p+\frac{1}{2}\zeta^{(1)}\right)}\frac{1}{\mathrm{ch}\left(p-\frac{1}{2}\zeta^{(1)}\right)}\frac{1}{\mathrm{ch}\left(p-\frac{1}{2}\zeta^{(1)}-\zeta'^{(2)}\right)}\\&\times\left(e^{\pi p}\frac{e^{2\pi i(\zeta^{(2)}+\zeta'^{(2)}+\zeta^{(1)})q}}{\mathrm{ch}\,q}e^{\pi p}+e^{-\pi p}\frac{e^{2\pi i(\zeta^{(2)}+\zeta'^{(2)}+\zeta^{(1)})q}}{\mathrm{ch}\,q}e^{-\pi p}\right)e^{-2\pi i\mu p}e^{\pi i\zeta^{(1)}q}\,.\end{aligned}\tag{2.128}$$

Applying the canonical transformation (2.118) and removing the exponential factors at the beginning and the end by conjugation, we obtain (up to an overall phase)[15] the density operator of the (second) mirror dual theory, which is a $U(2N)$ theory with two anti-symmetric hypermultiplets of masses $\widetilde{M}_1, \widetilde{M}_2$, four fundamental hypermultiplets of masses $\tilde{\mu}_\alpha$, $\alpha=1,\cdots,4$ and an FI parameter $\tilde{\zeta}$. The explicit

[15]Such phases are unphysical, see footnote 8.

mirror map between parameters is given by

$$
\begin{aligned}
&\widetilde{M}_1 = -\zeta^{(0)} - \zeta'^{(0)} - \zeta^{(1)}, \qquad \widetilde{M}_2 = -\zeta^{(2)} - \zeta'^{(2)} - \zeta^{(1)}, \\
&\tilde{\mu}_1 = \tfrac{1}{2}\zeta^{(1)} + \zeta'^{(0)}, \qquad \tilde{\mu}_2 = \tfrac{1}{2}\zeta^{(1)}, \qquad \tilde{\mu}_3 = -\tfrac{1}{2}\zeta^{(1)}, \qquad \tilde{\mu}_4 = -\tfrac{1}{2}\zeta^{(1)} - \zeta'^{(2)}, \\
&\quad \tilde{\zeta} = \mu\,.
\end{aligned}
\tag{2.129}
$$

Finally we may think of other ways to pair the $U(N)$ nodes of the $\hat{D}_4$-quiver which are similar to the two cases above. For instance we can consider the exchange $\zeta'^{(0)} \leftrightarrow \zeta'^{(2)}$ in (2.125). This symmetry, which is completely trivial on the $\hat{D}$-quiver side, manifests as a relation between two mirror $U(2N)$ theories which differ only by the values of their mass parameters

$$
\begin{aligned}
&\widetilde{M}_1 \to \widetilde{M}_1 + \tilde{\mu}_1 + \tilde{\mu}_4\,, \qquad \widetilde{M}_2 \to \widetilde{M}_2 - \tilde{\mu}_1 - \tilde{\mu}_4\,, \\
&\tilde{\mu}_1 \to -\tilde{\mu}_4, \qquad \tilde{\mu}_2 \to \tilde{\mu}_2, \qquad \tilde{\mu}_3 \to \tilde{\mu}_3, \qquad \tilde{\mu}_4 \to -\tilde{\mu}_1\,, \\
&\quad \tilde{\zeta} \to \tilde{\zeta}\,.
\end{aligned}
\tag{2.130}
$$

2.5.5 $\hat{D}_5$-quiver

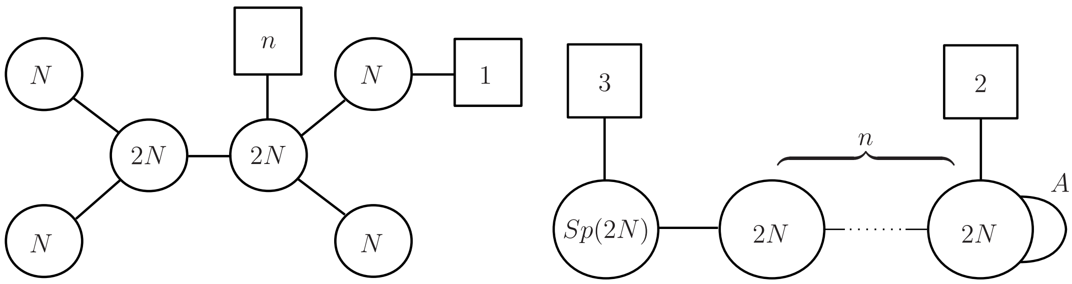

Figure 2.9: $\hat{D}_5$-quiver and its mirror dual theory.

The next example of a mirror map involves a $\hat{D}_5$-quiver with n fundamental hypermultiplets on one $U(2N)$ node and a single fundamental hypermultiplet on one $U(N)$ node as depicted in Figure 2.9. Shifting eigenvalues to set all bifundamental masses, and one of the $U(2N)$ fundamental masses to zero, we find the density operator (with obvious notations for the mass parameters)

$$
\begin{aligned}
\rho = {}& \frac{1}{4}\frac{1}{\operatorname{ch} p}\left(e^{2\pi i\zeta^{(0)}q}\frac{\operatorname{sh} p}{\operatorname{ch} p}e^{2\pi i\zeta'^{(0)}q} + e^{2\pi i\zeta'^{(0)}q}\frac{\operatorname{sh} p}{\operatorname{ch} p}e^{2\pi i\zeta^{(0)}q}\right)\frac{1}{\operatorname{ch} p}e^{2\pi i\zeta^{(1)}q} \\
&\times \frac{1}{\operatorname{ch} p}\frac{e^{2\pi i\zeta^{(2)}q}}{\operatorname{ch} q\prod_{\alpha=1}^{n-1}\operatorname{ch}\left(q+\mu_\alpha^{(2)}\right)}\frac{1}{\operatorname{ch} p}\left(\frac{e^{2\pi i\zeta^{(3)}q}}{\operatorname{ch}\left(q+\mu^{(3)}\right)}\frac{\operatorname{sh} p}{\operatorname{ch} p}e^{2\pi i\zeta'^{(3)}q}\right. \\
&\left. + e^{2\pi i\zeta'^{(3)}q}\frac{\operatorname{sh} p}{\operatorname{ch} p}\frac{e^{2\pi i\zeta^{(3)}q}}{\operatorname{ch}\left(q+\mu^{(3)}\right)}\right)\frac{1}{\operatorname{ch} p}\frac{e^{2\pi i\zeta^{(2)}q}}{\operatorname{ch} q\prod_{\alpha=1}^{n-1}\operatorname{ch}\left(q+\mu_\alpha^{(2)}\right)}\frac{1}{\operatorname{ch} p}e^{2\pi i\zeta^{(1)}q}\,.
\end{aligned}
\tag{2.131}
$$

Using the identities (2.120), (2.121) and (2.127), we can write the density operator as

$$
\begin{aligned}
\rho = &\frac{1}{2} e^{-2\pi i(\zeta^{(3)}+\zeta'^{(3)})\mu^{(3)}} \frac{1}{\operatorname{ch} p} e^{\pi i(\zeta^{(0)}+\zeta'^{(0)})q} \frac{\operatorname{sh} 2p}{\operatorname{ch}(p+\frac{1}{2}\zeta^{(0)}-\frac{1}{2}\zeta'^{(0)})\operatorname{ch}(p-\frac{1}{2}\zeta^{(0)}+\frac{1}{2}\zeta'^{(0)})} \\
&\times e^{\pi i(\zeta^{(0)}+\zeta'^{(0)})q} \frac{1}{\operatorname{ch} p} e^{2\pi i\zeta^{(1)}q} \frac{1}{\operatorname{ch} p} \frac{e^{2\pi i\zeta^{(2)}q}}{\operatorname{ch} q \prod_{\alpha=1}^{n-1} \operatorname{ch}\left(q+\mu_\alpha^{(2)}\right)} \frac{1}{\operatorname{ch} p} \\
&\times \frac{e^{2\pi i\mu^{(3)}p}}{\operatorname{ch}(p-\zeta'^{(3)})} \left(e^{\pi p} \frac{e^{2\pi i(\zeta^{(3)}+\zeta'^{(3)})q}}{\operatorname{ch} q} e^{\pi p} + e^{-\pi p} \frac{e^{2\pi i(\zeta^{(3)}+\zeta'^{(3)})q}}{\operatorname{ch} q} e^{-\pi p} \right) \frac{e^{-2\pi i\mu^{(3)}p}}{\operatorname{ch}(p+\zeta'^{(3)})} \\
&\times \frac{1}{\operatorname{ch} p} \frac{e^{2\pi i\zeta^{(2)}q}}{\operatorname{ch} q \prod_{\alpha=1}^{n-1} \operatorname{ch}\left(q+\mu_\alpha^{(2)}\right)} \frac{1}{\operatorname{ch} p} e^{2\pi i\zeta^{(1)}q} \,. \qquad (2.132)
\end{aligned}
$$

Further commuting exponential terms using (2.120) we obtain

$$
\begin{aligned}
\rho = &\frac{1}{2} e^{\pi i(\zeta^{(0)}+\zeta'^{(0)})q} \operatorname{ch}\left(p+\tilde{\mu}_3^{(0)}\right) \operatorname{ch} q \, e^{2\pi i\tilde{m}q} \, e^{-2\pi i(\zeta^{(3)}+\zeta'^{(3)})\mu^{(3)}} \\
&\times \frac{e^{-2\pi i\tilde{m}q}}{\operatorname{ch} q} \frac{\operatorname{sh} 2p}{\prod_{\beta=1}^{3} \operatorname{ch}\left(p+\tilde{\mu}_\beta^{(0)}\right)\operatorname{ch}\left(p-\tilde{\mu}_\beta^{(0)}\right)} \frac{e^{-2\pi i\tilde{m}q}}{\operatorname{ch} q} \left(\prod_{a=1}^{n-1} e^{2\pi i\tilde{\zeta}^{(a)}p} \frac{1}{\operatorname{ch} q} \right) \\
&\times \frac{e^{2\pi i\tilde{\zeta}^{(n)}p}}{\operatorname{ch} p \operatorname{ch}\left(p+\tilde{\mu}^{(n)}\right)} \left(e^{\pi p} \frac{e^{-2\pi i\widetilde{M}q}}{\operatorname{ch} q} e^{\pi p} + e^{-\pi p} \frac{e^{-2\pi i\widetilde{M}q}}{\operatorname{ch} q} e^{-\pi p} \right) \frac{e^{-2\pi i\tilde{\zeta}^{(n)}p}}{\operatorname{ch} p \operatorname{ch}\left(p-\tilde{\mu}^{(n)}\right)} \\
&\times \left(\prod_{a=1}^{n-1} \frac{1}{\operatorname{ch} q} e^{-2\pi i\tilde{\zeta}^{(n-a)}p} \right) \frac{e^{-2\pi i\tilde{m}q}}{\operatorname{ch} q} \frac{1}{\operatorname{ch}\left(p+\tilde{\mu}_3^{(0)}\right)} e^{-\pi i(\zeta^{(0)}+\zeta'^{(0)})q} ,
\end{aligned}
\qquad (2.133)
$$

with

$$
\begin{aligned}
&\tilde{m} = -\tfrac{1}{2}\zeta^{(0)} - \tfrac{1}{2}\zeta'^{(0)} - \zeta^{(1)} - \zeta^{(2)}, \qquad \widetilde{M} = -\zeta^{(3)} - \zeta'^{(3)} \\
&\tilde{\mu}_1^{(0)} = \tfrac{1}{2}\zeta^{(0)} - \tfrac{1}{2}\zeta'^{(0)}, \qquad \tilde{\mu}_2^{(0)} = \tfrac{1}{2}\zeta^{(0)} + \tfrac{1}{2}\zeta'^{(0)}, \qquad \tilde{\mu}_3^{(0)} = \tfrac{1}{2}\zeta^{(0)} + \tfrac{1}{2}\zeta'^{(0)} + \zeta^{(1)}, \\
&\tilde{\mu}^{(n)} = -\zeta'^{(3)}, \\
&\tilde{\zeta}^{(1)} = \mu_1^{(2)}, \qquad \tilde{\zeta}^{(n)} = \mu^{(3)} - \mu_{n-1}^{(2)}, \qquad \tilde{\zeta}^{(a)} = \mu_a^{(2)} - \mu_{a-1}^{(2)}, \quad a = 2, \cdots, n-1 \,.
\end{aligned}
\qquad (2.134)
$$

Applying the canonical transformation (2.118) and removing terms at the beginning and the end of (2.133) by conjugation, we obtain up to an overall phase (see footnote 8) the density operator of the mirror theory. This is a linear quiver with one $Sp(2N)$ node with three fundamental hypermultiplets, connected to n $U(2N)$ nodes, where the last $U(2N)$ node has one antisymmetric and two fundamental hypermultiplets. By shifting eigenvalues the masses of hypermultiplets transforming in the bifundamental representation of two $U(2N)$ nodes and the mass of one of the $U(2N)$ fundamental hypermultiplets can be set to zero. This leaves us with the masses: $\tilde{m}$ for the $Sp(2N) \times U(2N)$ bifundamental hypermultiplet, $\widetilde{M}$ for the antisymmetric hypermultiplet, $\tilde{\mu}_{1,2,3}^{(0)}$ for the three $Sp(2N)$ fundamental hypermultiplets, $\tilde{\mu}^{(n)}$ for one $U(2N)$ fundamental hypermultiplet. Moreover the theory has FI parameters $\tilde{\zeta}^{(a)}$, for $a = 1, \cdots, n$. The mirror map between parameters is given by (2.134).

This general approach to finding mirror maps can be easily extended to more general $\hat{D}$-quivers.

2.5.6 Mirrors involving 'bad' quivers

For mirrors involving a 'bad' linear quiver, proposed in [81], the matrix model of the linear quiver is divergent,[16] while the matrix model of the $\hat{D}$ quiver is still finite. This of course means that the corresponding density operators will be inequivalent, since one will be of trace class and the other not. Nevertheless, it is still interesting to compare the density operators we find on either side of the proposed duality, and we do so be studying their Wigner transforms (for definitions of Wigner phase space see section 2.6.1).

One example considered in [81] for instance is the naive duality between a $\hat{D}_4$-quiver with two fundamental hypermultiplets on an external $U(N)$ node, and an $Sp(2N)^2$ linear quiver with four fundamental hypermultiplets on one $Sp(2N)$ node, and zero on the other. Manipulations as in (2.166) give the Wigner transformed density operator of the $\hat{D}_4$-quiver as

$$\frac{1}{\mathrm{ch}^2 p} \star \frac{\mathrm{sh}\, 2p}{\mathrm{ch}^2 q} \star \frac{\mathrm{sh}\, 2p}{\mathrm{ch}^6 p}\,. \tag{2.135}$$

Meanwhile the Wigner transformed density operator of the linear quiver is

$$\frac{1}{\mathrm{ch}\, p} \star \mathrm{sh}\, 2q \star \frac{1}{\mathrm{ch}\, p} \star \frac{\mathrm{sh}\, 2q}{\mathrm{ch}^8 q}\,. \tag{2.136}$$

These operators would be related by the canonical transformation (2.118) (and conjugation by $\frac{1}{\mathrm{ch}^2 p}$), if we also included the replacement (with $n = 1$ and $m = 0$)

$$\frac{1}{\mathrm{ch}^n p} \star \frac{\mathrm{sh}\, 2q}{\mathrm{ch}^m p} \star \frac{1}{\mathrm{ch}^n p} \to \frac{\mathrm{sh}\, 2q}{\mathrm{ch}^{2n+m} p} \tag{2.137}$$

This is an *ad hoc* regularisation of the partition function, transforming a non trace class density operator to one that is trace class, based on the assumption that the regularised partition function should satisfy the naive mirror symmetry. In fact, the replacement (2.137) (with suitable n, m) easily regularises and identifies a mirror dual for any bad linear quiver with no fundamental hypermultiplets on one

[16] It was proposed in [78] that the divergence of 'bad' quiver matrix model is related to a mismatch between the R-symmetry group in the UV localisation computation, and the R-symmetry group at the infrared fixed point.

or both terminal nodes, provided the theory has in total at least four fundamental hypermultiplets.

It would be interesting to understand if (2.137) can be derived from a proper regularisation of the divergent integrals in the matrix model.

2.6 Evaluating the partition function at large N

Having re-expressed the S^3 partition functions of $\hat{A}$ and $\hat{D}$ quiver theories as free fermion partition functions, we proceed now with their evaluation following the technique developed in [14]. More precisely we compute the perturbative part in the large μ expansion of the grand potential $J(\mu)$ and extract from it the perturbative part in the large N expansion of the partition function $Z(N)$, which turns out to be the Airy function. Our strategy to compute $J(\mu)$ has new ingredients compared to [14], in particular we use a simplifying recursion method to extract the perturbative expansion of the spectral traces $Z_l = \operatorname{Tr} \rho^l$, for arbitrary $l \geqslant 1$. This method can be used for any density operator ρ. We also rely on the integral representation of $J(\mu)$, following [85], to evaluate its perturbative part. Mass, CS and FI parameters introduce extra difficulties in the computation and we set all of them to zero in this section.

2.6.1 General strategy

The starting point for our computation are the Fermi gas formulations of $\hat{A}$ (2.65) and $\hat{D}$ (2.81) partition functions. Both are written as a sum over S_N permutations, but differ for $\hat{D}$ quivers by the inclusion of an additional factor of 2^{n_σ}, where n_σ counts the number of cycles in σ.

$$Z(N) = \frac{1}{N!} \sum_{\sigma \in S_N} \frac{(-1)^\sigma}{2^{n_\sigma}} \int \prod_{i=1}^{N} d\lambda_i \prod_{i=1}^{N} \rho(\lambda_i, \lambda_{\sigma(i)}) \,, \tag{2.138}$$

The standard analysis [65] is to factor the integral into its closed loops, which give spectral traces Z_l

$$\int d\lambda_1 \cdots d\lambda_l \, \rho(\lambda_1, \lambda_2)\rho(\lambda_2, \lambda_3) \cdots \rho(\lambda_l, \lambda_1) = \operatorname{Tr} \rho^l \equiv Z_l \,. \tag{2.139}$$

These loops correspond to the cycles of the permutation σ, and so the summand of (2.138) depends only on Z_l and the conjugacy class of σ. Conjugacy classes of S_N can be labelled a set of integers $\{m_l\}$, where m_l is the number of cycles of length l.

In terms of this labelling we have

$$\frac{1}{2^{n_\sigma}} = \prod_l \frac{1}{2^{m_l}}\,, \tag{2.140}$$

and the number of permutations in a given conjugacy class is given by

$$\frac{N!}{\prod_l m_l!\, l^{m_l}}\,. \tag{2.141}$$

With these combinatorics (2.138) becomes

$$Z(N) = \sum_{\{m_l\}}{}' \prod_l \frac{(\frac{1}{2}Z_l)^{m_l}\,(-1)^{(l-1)m_l}}{m_l!\, l^{m_l}}\,, \tag{2.142}$$

where the primed sum denotes a sum over sets that satisfy $\sum_l l m_l = N$.

The computation of $Z(N)$ thus boils down to the evaluation of Z_l. To compute results for large N, the standard approach [65] is to consider the grand canonical partition function

$$\Xi(z) = 1 + \sum_{N=1}^{\infty} Z(N)\kappa^N = e^{J(\mu)}, \qquad \kappa = e^{\mu}\,, \tag{2.143}$$

where μ is the chemical potential, and $J(\mu)$ the grand canonical potential, given by

$$J(\mu) = -\sum_{l=1}^{\infty} \frac{(-1)^l Z_l e^{\mu l}}{2l}\,. \tag{2.144}$$

The strategy is to first find an expression for Z_l, then to resum the expression and obtain $J(\mu)$ using (2.144), and finally to recover $Z(N)$ by computing

$$Z(N) = \frac{1}{2\pi i}\int_{\mu_0-\pi i}^{\mu_0+\pi i} d\mu\, e^{J(\mu)-\mu N}\,, \tag{2.145}$$

where μ_0 can be chosen arbitrarily without affecting the result.

In practice, computing Z_l exactly for arbitrary l is highly non trivial. To make the problem tractable it is useful to reformulate it within Wigner's phase space [14]. For a general review of Wigner's phase space see [86]; here we simply summarise the properties that we require.

The Wigner transform of an operator[17] $\hat{A}$ is given (with $\hbar = \frac{1}{2\pi}$) by

$$A_W(q,p) = \int dq' \left\langle q - \frac{q'}{2}\right| \hat{A} \left| q + \frac{q'}{2}\right\rangle e^{2\pi i p q'}\,. \tag{2.146}$$

[17] To avoid confusion between phase space variables p, q, and the canonical position and momentum operators, we give all operators hats for the remainder of this section.

Some important identities are

$$(\hat{A}\hat{B})_W = A_W \star B_W , \quad \star = \exp\left[\frac{i}{4\pi}\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overrightarrow{\partial}_q\overleftarrow{\partial}_p\right)\right], \quad \mathrm{Tr}(\hat{A}) = \int dpdq A_W \, . \tag{2.147}$$

In the language of phase space Z_l becomes

$$Z_l = \int dpdq \, \overbrace{\rho_W \star \rho_W \cdots \star \rho_W}^{l} \, . \tag{2.148}$$

We generate an expansion for the integrand of (2.148) by performing a derivative expansion of the star products (2.147). To this end, we introduce into the star product an expansion parameter ϵ which will be set at the end to 1.[18]

$$\begin{aligned} \star &= \exp\left[\frac{i\epsilon}{4\pi}\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overrightarrow{\partial}_q\overleftarrow{\partial}_p\right)\right] \\ &= 1 + \frac{i\epsilon}{4\pi}\left(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overrightarrow{\partial}_q\overleftarrow{\partial}_p\right) - \frac{\epsilon^2}{32\pi^2}\left(\overleftarrow{\partial}^2_q\overrightarrow{\partial}^2_p + \overrightarrow{\partial}^2_q\overleftarrow{\partial}^2_p - 2\overleftarrow{\partial}_{q,p}\overrightarrow{\partial}_{q,p}\right) + \mathcal{O}(\epsilon^3) \, . \end{aligned} \tag{2.150}$$

In [14], the role of the expansion parameter ϵ was played by the Planck constant $\hbar$, which was proportional to the Chern-Simons level of the ABJM theory. We do not have a tunable $\hbar$ here, however it still proves useful to consider the derivative expansion associated to ϵ, as we now explain.

The ϵ expansion of the integrand of (2.148) takes the form

$$(\hat{\rho}^l)_W(p,q) = \sum_{n\geqslant 0} \epsilon^n \, \rho_{l(n)}(p,q). \tag{2.151}$$

Note that the ϵ factors come from the expansion of the star products present in (2.148), as well as those arising when replacing the density operator (2.98) by its Wigner transform. Z_l can then be evaluated order by order in ϵ

$$Z_l = \sum_{n\geqslant 0} \epsilon^n Z_{l(n)}, \qquad Z_{l(n)} = \int dpdq \, \rho_{l(n)} \, . \tag{2.152}$$

[18] In addition to the differential expression for the star product (2.150), it also has an equivalent integral representation

$$(f \star g)(p,q) = \frac{4}{\epsilon^2}\int dq'dp'dq''dp'' f(p+p', q+q')g(p+p'', q+q'')e^{4\pi i/\epsilon(q'p''-q''p')} \, . \tag{2.149}$$

In some cases (for instance when f or g involve delta functions) (2.149) produces an exact result that is non perturbative in ϵ [87]. In these cases extra care should be taken, because the perturbative star product (2.147) is not valid.

Resumming each term using (2.144) then generates an ϵ expansion for $J(\mu)$

$$J(\mu) = \sum_{n \geqslant 0} \epsilon^n J_{(n)}(\mu), \qquad J_{(n)}(\mu) = -\sum_{l=1}^{\infty} \frac{(-1)^l Z_{l(n)} e^{\mu l}}{2l} . \tag{2.153}$$

As was found for $\hat{A}$-quivers in [14], we anticipate that for $\hat{D}$-quivers (and their linear mirrors), $J(\mu)$ also admits an asymptotic expansion of the form

$$J(\mu) = \frac{C(\epsilon)}{3}\mu^3 + B(\epsilon)\mu + A(\epsilon) + \mathcal{O}(e^{-\alpha\mu}) , \qquad \alpha > 0 , \tag{2.154}$$

where each of the coefficients A, B and C are given by power expansions in ϵ. In principle, to obtain a meaningful result for A, B and C we should now compute and then resum an infinite series of corrections in powers of ϵ, which are really all of the same order since $\epsilon = 1$. From the study of $\hat{A}$-quivers, it is expected that the expansions of $C(\epsilon)$ and $B(\epsilon)$ truncate at orders ϵ^0 and ϵ^2 respectively, so that the first few orders of the ϵ expansion are sufficient to compute them exactly. We give a proof of this truncation for $\hat{D}$-quivers with equal number of fundamental hypermultiplets on each pair of terminating $U(N)$ nodes in appendix D. We assume that it holds for the other quivers as well.[19]

It remains to plug the result (2.154) into (2.145) to extract the perturbative part of Z at large N. In practice the evaluation is done by setting the contour integral parameter μ_0 to the saddle point μ^* of the integrand and by extending the contour along all the imaginary axis to $(\mu^* - i\infty, \mu^* + i\infty)$. As explained in [14, 19], this change of contour does not affect the perturbative part of the result. The integration leads to the Airy function behaviour of the partition function at large N, which is our main result

$$Z(N) = C^{-\frac{1}{3}} e^A \operatorname{Ai}\left[C^{-\frac{1}{3}}(N - B)\right] + Z_{\text{np}}(N) , \tag{2.155}$$

where $Z_{\text{np}}(N)$ denotes non-perturbative, exponentially suppressed contributions, and we note that the undetermined coefficient A only affects the overall prefactor.

2.6.2 Recursive formula for $(\hat{\rho}^l)_W$

In this subsection we present a simple recursive approach for evaluating the coefficients in the ϵ expansion of $(\hat{\rho}^l)_W$ (2.151). This comes from the ϵ expansion of

$$(\hat{\rho}^{l+1})_W = (\hat{\rho}^l)_W \star \rho_W . \tag{2.156}$$

[19] We checked that C and B do not receive contributions at order ϵ^4 for all $\hat{D}$-quivers.

One first needs to evaluate the ϵ expansion of ρ_W (which is due to replacing all operator products by star products)

$$\rho_W(p,q) = \sum_{n\geqslant 0} \epsilon^n \, \rho_{(n)}(p,q) \,, \tag{2.157}$$

which also serves as the initial conditions for the recursion $\rho_{1(n)} = \rho_{(n)}$.

At order ϵ^0, equation (2.156) gives

$$\rho_{l+1(0)} = \rho_{l(0)}\rho_{(0)} \,, \tag{2.158}$$

which is solved by

$$\rho_{l(0)} = \rho_{(0)}^l \,. \tag{2.159}$$

At order ϵ^1 we get

$$\rho_{l+1(1)} = \rho_{(0)}^l \rho_{(1)} + \rho_{(0)}\rho_{l(1)} \qquad \Rightarrow \qquad \rho_{l(1)} = l\rho_{(0)}^{l-1}\rho_{(1)} \,. \tag{2.160}$$

At order ϵ^2 we then get

$$\begin{aligned}\rho_{l+1(2)} = \rho_{(0)}\rho_{l(2)} + \rho_{(0)}^l\rho_{(2)} + l\rho_{(0)}^{l-1}\rho_{(1)}^2 - \frac{1}{32\pi^2}\rho_{(0)}^{l-2} l\Big[2\big(\rho_{(0)}\rho''_{(0)}\ddot{\rho}_{(0)} - \rho_{(0)}\dot{\rho}'^2_{(0)}\big) \\ + (l-1)\big(\ddot{\rho}_{(0)}\rho'^2_{(0)} + \rho''_{(0)}\dot{\rho}^2_{(0)} - 2\rho'_{(0)}\dot{\rho}_{(0)}\dot{\rho}'_{(0)}\big)\Big],\end{aligned} \tag{2.161}$$

where

$$\dot{f}(p,q) \equiv \partial_p \, f(p,q) \,, \qquad f'(p,q) \equiv \partial_q \, f(p,q) \,. \tag{2.162}$$

Solving the recurrence relation, with initial condition $\rho_{1(2)} = \rho_{(2)}$ yields

$$\begin{aligned}\rho_{l(2)} = l\rho_{(0)}^{l-1}\rho_{(2)} + \frac{1}{2}l(l-1)\rho_{(0)}^{l-2}\rho_{(1)}^2 - \frac{1}{96\pi^2}\rho_{(0)}^{l-3} l(l-1)\Big[3\big(\rho_{(0)}\rho''_{(0)}\ddot{\rho}_{(0)} - \rho_{(0)}\dot{\rho}'^2_{(0)}\big) \\ + (l-2)\big(\ddot{\rho}_{(0)}\rho'^2_{(0)} + \rho''_{(0)}\dot{\rho}^2_{(0)} - 2\rho'_{(0)}\dot{\rho}_{(0)}\dot{\rho}'_{(0)}\big)\Big].\end{aligned} \tag{2.163}$$

This procedure can be straightforwardly continued to higher order in ϵ. Finally we plug the expansion coefficients into (2.152) to obtain the ϵ expansion of Z_l. In particular this gives

$$\begin{aligned}Z_{l(0)} &= \int dpdq \, \rho_{(0)}^l \,, \\ Z_{l(1)} &= \int dpdq \, l\rho_{(0)}^{l-1}\rho_{(1)} \,, \\ Z_{l(2)} &= \int dpdq \, \left(l\rho_{(0)}^{l-1}\rho_{(2)} + \frac{1}{2}l(l-1)\rho_{(0)}^{l-2}\rho_{(1)}^2 - \frac{1}{96\pi^2}l^2(l-1)(l-2)\rho_{(0)}^{l-4}\dot{\rho}^2_{(0)}\rho'^2_{(0)} \right),\end{aligned} \tag{2.164}$$

where the last line follows from integrating (2.163) by parts, and using the fact that for $\hat{A}$ or $\hat{D}$ quivers we always have the decomposition $\rho_{(0)} = t(p)u(q)$.

We stress that this algorithm is very general and can be applied to any $\hat{A}$ or $\hat{D}$-quiver. All one needs in order to compute Z_l is to plug into (2.164) the ϵ expansion of the density operator itself.

2.6.3 Computing $Z(N)$ for $\hat{D}$ quivers

We now show how to apply the approach outlined in the previous sections and compute $Z(N)$ for a generic $\hat{D}$-quiver with an arbitrary number of nodes and an arbitrary number of fundamental hypermultiplets on each node, but no mass, CS or FI terms. The quiver diagram is shown in figure 2.2, and the density operator is given by (2.98) with

$$F^{(a)}(q) = \frac{1}{\mathrm{ch}^{n^{(a)}} q}, \qquad F'^{(a)}(q) = \frac{1}{\mathrm{ch}^{n'^{(a)}} q}\,. \tag{2.165}$$

We first work out the ϵ-expansion of ρ_W itself, which is then plugged in to the result of the recursive formula (2.164).

Using manipulations similar to those used in section 2.5.3

$$\begin{aligned}
&\frac{1}{\mathrm{ch}^{n^{(0)}} q} \star \frac{\mathrm{sh}\, p}{\mathrm{ch}\, p} \star \frac{1}{\mathrm{ch}^{n'^{(0)}} q} + \frac{1}{\mathrm{ch}^{n'^{(0)}} q} \star \frac{\mathrm{sh}\, p}{\mathrm{ch}\, p} \star \frac{1}{\mathrm{ch}^{n^{(0)}} q} \\
&= \frac{1}{\mathrm{ch}^{\min(n^{(0)},n'^{(0)})} q} \star \frac{1}{\mathrm{ch}\, p} \star \left(\mathrm{sh}\, p \star \frac{1}{\mathrm{ch}^{|n^{(0)}-n'^{(0)}|} q} \star \mathrm{ch}\, p + \mathrm{ch}\, p \star \frac{1}{\mathrm{ch}^{|n^{(0)}-n'^{(0)}|} q} \star \mathrm{sh}\, p \right) \\
&\quad \star \frac{1}{\mathrm{ch}\, p} \star \frac{1}{\mathrm{ch}^{\min(n^{(0)},n'^{(0)})} q} \\
&= \frac{1}{\mathrm{ch}^{\min(n^{(0)},n'^{(0)})} q} \star \frac{1}{\mathrm{ch}\, p} \star \frac{2\,\mathrm{sh}\, 2p}{\mathrm{ch}^{|n^{(0)}-n'^{(0)}|} q} \star \frac{1}{\mathrm{ch}\, p} \star \frac{1}{\mathrm{ch}^{\min(n^{(0)},n'^{(0)})} q}\,,
\end{aligned} \tag{2.166}$$

where in the last step we evaluated the expression inside the parentheses using the exact star product (2.149). This allows us to write the Wigner transform of the density operator (2.98) as[20]

$$\begin{aligned}
\rho_W = {} & e^{T(p)} \star e^{U_1(q)} \star e^{T(p)} \star e^{S(p)+2U_0(q)} \star e^{T(p)} \star \left(\prod_{k=1}^{\lambda-1}{}_\star\, e^{U_k(q)} \star e^{T(p)} \right) \\
& \star e^{S(p)+2U_\lambda(q)} \star \left(\prod_{k=1}^{\lambda-2}{}_\star\, e^{T(p)} \star e^{U_{\lambda-k}(q)} \right),
\end{aligned} \tag{2.167}$$

[20] $\prod_\star$ is defined by ordered star multiplication

where

$$S(p) = \log \operatorname{sh} 2p, \qquad T(p) = \log \frac{1}{\operatorname{ch} p}, \qquad U_i(q) = \log \frac{1}{\operatorname{ch}^{\eta_i} q}, \tag{2.168}$$

and

$$\begin{aligned} \eta_0 &= \frac{1}{2}|n^{(0)} - n'^{(0)}|, \qquad \eta_1 = \min(n^{(0)}, n'^{(0)}), \qquad \eta_i = n^{(i-1)}, \quad i = 2, \cdots, L \\ \eta_\lambda &= \frac{1}{2}|n^{(L)} - n'^{(L)}|, \qquad \eta_{\lambda-1} = \min(n^{(L)}, n'^{(L)}), \qquad \lambda = L + 2. \end{aligned} \tag{2.169}$$

A first useful manipulation is to conjugate[21] the density operator into a more symmetric form that resembles a palindrome

$$\begin{aligned} \rho_W \approx \sqrt[\star]{e^{S(p)+2U_\lambda(q)}} \star e^{T(p)} \star \left(\prod_{k=1}^{\lambda-1}{}_\star\, e^{U_{\lambda-k}(q)} \star e^{T(p)} \right) \star e^{S(p)+2U_0(q)} \\ \star \left(\prod_{k=1}^{\lambda-1}{}_\star\, e^{T(p)} \star e^{U_k(q)} \right) \star e^{T(p)} \star \sqrt[\star]{e^{S(p)+2U_\lambda(q)}}, \end{aligned} \tag{2.170}$$

where $\sqrt[\star]{\ }$ is the star square root, and its expansion gives

$$\sqrt[\star]{e^{S(p)+2U_\lambda(q)}} = e^{\frac{1}{2}S+U_\lambda} \left(1 + \frac{\epsilon^2}{128\pi^2} \big(2\ddot{S}U''_\lambda + 2\ddot{S}U'^2_\lambda + \dot{S}^2 U''_\lambda \big) + \mathcal{O}(\epsilon^4) \right). \tag{2.171}$$

The reason for making this conjugation into palindromic form is that the ϵ expansion of (2.170) is purely in even powers of ϵ. The easiest way then to compute the ϵ expansion is to build it up from the central $e^{S(p)+2U_0(q)}$ piece. We first compute

$$e^T \star e^{S+2U_0} \star e^T = e^{2T+S+2U_0} \left(1 - \frac{\epsilon^2}{8\pi^2} \ddot{T} \big(U''_0 + 2U'^2_0 \big) + \mathcal{O}(\epsilon^4) \right). \tag{2.172}$$

By plugging this result in, we can then easily compute

$$\begin{aligned} &e^{U_1} \star \left(e^T \star e^{S+2U_0} \star e^T \right) \star e^{U_1} \\ &\quad = e^{2T+S+2U_0+2U_1} \left(1 - \frac{\epsilon^2}{8\pi^2} \ddot{T} (U''_0 + 2U'^2_0) - \frac{\epsilon^2}{16\pi^2} U''_1 \big(2\ddot{T} + \ddot{S} + (2\dot{T} + \dot{S})^2 \big) + \mathcal{O}(\epsilon^4) \right). \end{aligned} \tag{2.173}$$

[21] Since we are ultimately computing $Z_l = \operatorname{Tr} \hat{\rho}^l$ we can use cyclicity of the trace to make conjugations $\hat{\rho} \to \hat{V}^{-1} \hat{\rho} \hat{V}$, which in the language of phase space becomes $\rho_W \to (\hat{V}^{-1})_W \star \rho_W \star V_W$

Continuing this procedure we find the full expansion of (2.170) up to $\mathcal{O}(\epsilon^2)$

$$
\begin{aligned}
\rho_W = e^{2\lambda T+2S+2\sum_{k=0}^{\lambda}U_k}\Bigg[1 &- \frac{\epsilon^2}{8\pi^2}\ddot{T}\sum_{k=0}^{\lambda-1}\left(\sum_{j=0}^{k}U_j'' + 2\Big(\sum_{j=0}^{k}U_j'\Big)^2\right)\\
&- \frac{\epsilon^2}{16\pi^2}\sum_{k=1}^{\lambda}U_k''\left(\ddot{S}+2k\ddot{T}+\big(\dot{S}+2k\dot{T}\big)^2\right)\\
&- \frac{\epsilon^2}{16\pi^2}\left(\ddot{S}\Big(\sum_{k=0}^{\lambda-1}U_k''+2\sum_{k=0}^{\lambda-1}U_k'\sum_{j=0}^{\lambda}U_j'\Big)+U_\lambda''\dot{S}\big(\dot{S}+2\lambda\dot{T}\big)\right)+\mathcal{O}(\epsilon^4)\Bigg].
\end{aligned}
\tag{2.174}
$$

The coefficients of this expansion are $\rho_{(0)}$ and $\rho_{(2)}$ (2.157), which serve as the seed for the recursion and can be plugged directly into (2.164). We find that $Z_{l(0)}$ is given by

$$
Z_{l(0)} = \int dpdq\,\rho_{(0)}^l = \int dpdq\,\frac{\mathrm{th}^{2l}\,p}{\mathrm{ch}^{2l(\lambda-2)}\,p\,\mathrm{ch}^{2l\nu}\,q}, \qquad \nu = \sum_{k=0}^{\lambda}\eta_k\,. \tag{2.175}
$$

The expression for $Z_{l(2)}$ is considerably more involved, but it can be simplified by integrating by parts to remove all double derivatives. Integrating by parts the first term in $Z_{l(2)}$ (2.164) gives

$$
\begin{aligned}
&\int dpdq\,l\rho_{(0)}^{l-1}\rho_{(2)}\\
&= \frac{1}{4\pi^2}\int dpdq\,l^2 e^{2l(\lambda T+S+\sum_{k=0}^{\lambda}U_k)}\Bigg[2\dot{T}\big(\dot{S}+\lambda\dot{T}\big)\sum_{k=0}^{\lambda-1}\sum_{j=0}^{k}U_j'\Big(\sum_{i=0}^{k}U_i' - l\sum_{i=0}^{\lambda}U_i'\Big)\\
&\quad+\frac{1}{2}\sum_{k=1}^{\lambda}U_k'\big(\dot{S}+2k\dot{T}\big)\big(\dot{S}+2k\dot{T}-2l\big(\dot{S}+\lambda\dot{T}\big)\big)\sum_{j=0}^{\lambda}U_j'\\
&\quad-\left(\dot{S}\big(\dot{S}+\lambda\dot{T}\big)(l-1)\sum_{k=0}^{\lambda-1}U_k' - \frac{1}{2}\dot{S}\big(\dot{S}+2\lambda\dot{T}\big)U_\lambda'\right)\sum_{j=0}^{\lambda}U_j'\Bigg].
\end{aligned}
\tag{2.176}
$$

The second (non zero) term in $Z_{l(2)}$ gives

$$
\begin{aligned}
&\int dpdq\,\frac{-1}{96\pi^2}l^2(l-1)(l-2)\rho_{(0)}^{l-4}\dot{\rho}_{(0)}^2\rho_{(0)}'^2\\
&\quad= \int dpdq\,\frac{-1}{6\pi^2}l^2(l-1)(l-2)e^{2l(\lambda T+S+\sum_{k=0}^{\lambda}U_k)}\left(\sum_{j=0}^{k}\dot{U}_j\right)^2(\dot{S}+\lambda\dot{T})^2\,.
\end{aligned}
\tag{2.177}
$$

Combining these expressions, substituting (2.168) and simplifying leads finally to

$$
\begin{aligned}
Z_{l(2)} = \int dpdq\, l^2 \frac{\pi^2}{24} \frac{\operatorname{th}^{2l-2} p \operatorname{th}^2 q}{\operatorname{ch}^{2l(\lambda-2)} p \operatorname{ch}^{2l\nu} q} \Big[& 3(2l-1)\Delta\nu - (4l^2-1)\nu^2 \\
& + 2\operatorname{th}^2 p \Big((4l^2-1)(\lambda-1)\nu^2 - 3\Delta\nu(l(\lambda-2)+1) - 6\Sigma_1 \Big) \\
& - \operatorname{th}^4 p \Big((4l^2-1)(\lambda-1)^2\nu^2 - 3\lambda^2\nu^2 + 3\Delta\nu(2l(\lambda-1)+1) - 12(\Sigma_2-\Sigma_1) \Big) \Big],
\end{aligned}
\tag{2.178}
$$

where

$$
\begin{aligned}
\Delta &= \eta_0 + \eta_\lambda, \qquad \Sigma_1 = \sum_{i>j} \eta_i \eta_j (i-j), \\
\Sigma_2 &= \sum_{i>j} \eta_i \eta_j \left((\lambda-i)^2 + j^2 + \lambda^2 \right) - \sum_{i=0}^{\lambda} \eta_i^2 i(\lambda-i).
\end{aligned}
\tag{2.179}
$$

In order to evaluate the integrals appearing in $Z_{l(0)}$ and $Z_{l(2)}$, we require only the identity

$$
\int dx \frac{\operatorname{th}^a x}{\operatorname{ch}^b x} = \frac{(1+e^{i\pi a})\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b}{2}\right)}{2^{b+1}\pi\Gamma\left(\frac{a+b+1}{2}\right)}, \qquad \mathrm{Re}(a) > -1, \quad \mathrm{Re}(b) > 0 \tag{2.180}
$$

Integrating (2.175) and (2.178) using (2.180), simplifying by shifting the arguments of Gamma functions and choosing to express the result in terms of $L = \lambda - 2$ we find

$$
\begin{aligned}
Z_{l(0)} &= \frac{1}{2^{2l(\nu+L)}\pi^{3/2}} \frac{\Gamma(l+\frac{1}{2})\Gamma(\nu l)\Gamma(Ll)}{\Gamma(\nu l+\frac{1}{2})\Gamma((L+1)l+\frac{1}{2})}, \\
Z_{l(2)} &= \frac{\pi^{1/2}}{3\cdot 2^{2l(\nu+L)+3}} l^2 F(l) \frac{\Gamma(l+\frac{1}{2})\Gamma(\nu l)\Gamma(Ll)}{\Gamma(\nu l+\frac{3}{2})\Gamma((L+1)l+\frac{3}{2})},
\end{aligned}
\tag{2.181}
$$

where

$$
\begin{aligned}
F(l) = {} & \nu^2(1+(L+1)(L+2)) - 3\Delta\nu - 3(\Sigma_1+\Sigma_2) \\
& + l\big(\nu^2(L+2)(L+3) - 6\Delta\nu(L+1) - 6(L+1)\Sigma_1 + 6\Sigma_2\big) - 2l^2\nu^2 L(L+1).
\end{aligned}
\tag{2.182}
$$

Having computed the ϵ expansion of Z_l, we can now compute the ϵ expansion for $J(\mu)$ by resuming each term using (2.153). As for the $\hat{A}$ quivers, this gives some complicated hypergeometric functions [14, 33], from which we can then extract the large μ asymptotic expansion (2.154). Ultimately, we are interested only in the perturbative part of this expansion. A recent paper [85] pointed out a very elegant and simple way to extract this perturbative piece of $J(\mu)$, just by evaluating a single residue involving Z_l.

The first step is to write a Mellin-Barnes integral representation for the infinite sum (2.153)

$$J_{(n)}(\mu) = -\sum_{l=1}^{\infty} \frac{(-1)^l Z_{l(n)} e^{\mu l}}{2l} = -\int_{c-i\infty}^{c+i\infty} \frac{dl}{4\pi i} \Gamma(l)\Gamma(-l) Z_{l(n)} e^{l\mu}\,, \tag{2.183}$$

where c can be chosen arbitrarily in $(0,1)$, and $Z_{l(n)}$ should now be regarded as a function of l, analytically continued to the complex plane. To see how this integral representation reproduces the infinite sum (2.183), note that for $\mu < 0$ we can close the contour of integration around the region with $\mathrm{Re}(l) > c$. Since $Z_{l(n)}$ itself has no poles in this region, the only enclosed poles are the simple poles of $\Gamma(-l)$ for $l = n \in \mathbb{N}^+$. Using the fact that

$$\operatorname*{Res}_{l=n} \Gamma(-l) = \frac{(-1)^n}{n!}\,, \tag{2.184}$$

we recover (2.183).

Since we are interested in the asymptotic region $\mu \gg 0$, we close the contour of integration around the region $\mathrm{Re}(l) < c$. In this region there can be poles due to both $Z_{l(n)}$ and $\Gamma(l)$. The residue at $l = 0$ turns out to be the only one giving a contribution that is not exponentially suppressed at large μ. Therefore we can immediately evaluate

$$J_{(n)}(\mu) = -\frac{1}{2} \operatorname*{Res}_{l=0} \Gamma(l)\Gamma(-l) Z_{l(n)} e^{l\mu} + \mathcal{O}(e^{-\alpha\mu}), \qquad \alpha > 0\,. \tag{2.185}$$

Evaluating in this way the perturbative contributions to $J(\mu)$ from $J_{(0)}(\mu)$ and $J_{(2)}(\mu)$, we find the asymptotic expansion of the form (2.154), with C and B coefficients

$$C = \frac{1}{4\pi^2 L\nu}, \qquad B = \frac{1 - 3\Delta\nu + 2\nu^2 + L(3+L)(\nu^2-1) - 3(\Sigma_1 + \Sigma_2)}{12L\nu}\,. \tag{2.186}$$

As explained before, the coefficients C and B do not receive contributions from higher order terms in the ϵ expansion, so that (2.186) provides the exact result. These coefficients characterise the full perturbative part of $Z(N)$ as an Airy function (2.155), up to the overall coefficient A which is undetermined by our analysis.

2.6.4 Computing $Z(N)$ for $\hat{A}$ quivers

For completeness we use our recursive method also to compute $Z(N)$ for $\hat{A}$ quivers with aribtrary number of nodes and fundamental hypermultiplets for each node, but with all mass, FI and CS parameters set to zero. The quiver diagram is shown in Figure2.1. The results we obtain are not new, since, as discussed in section 2.5.2, these theories are mirror dual to $\hat{A}$ quiver theories which have have no fundamental hypermultiplets but CS terms $+1$, 0 or -1. Theories of this class were considered in [33], and indeed their results for C and B agree with ours after applying the mirror map.

The density operator for the theory we have described is given by (2.66) with

$$F^{(a)}(q) = \frac{1}{\mathrm{ch}^{n^{(a)}} q}\,, \qquad m^{(a)} = 0\,. \tag{2.187}$$

Its Wigner transform can be written as

$$\rho_W = \prod_{a=1}^{L}{}_\star e^{U_a(q)} \star e^{T(p)}\,, \tag{2.188}$$

where

$$U_a(q) = n^{(a)} \log \frac{1}{\mathrm{ch}\, q}, \qquad T(p) = \log \frac{1}{\mathrm{ch}\, p}\,. \tag{2.189}$$

Following our prescription we first work out the ϵ-expansion of ρ_W itself, which we can then plug into the recursive formula (2.164). From (2.150) the expansion of (2.188) up to $\mathcal{O}(\epsilon)$ is readily obtained

$$\rho_W = e^{LT+\sum_{a=1}^L U_a}\left[1 + \frac{i\epsilon}{4\pi}\sum_{a=1}^{L} \dot{T} U_a'(L-2a+2) + \mathcal{O}(\epsilon^2)\right] \tag{2.190}$$

To push the expansion up to ϵ^2 it is useful to consider each star product sequentially, starting with the leftmost

$$e^{U_0} \star e^T = e^{T+U_0}\left[1 + \frac{i\epsilon}{4\pi}\dot{T}U_0' - \frac{\epsilon^2}{32\pi^2}(\ddot{T}+\dot{T}^2)(U_0''+U_0'^2) + \mathcal{O}(\epsilon^3)\right]. \tag{2.191}$$

Next we should expand the star product with e^{U_1}, and so on. In general the star product with e^{U_k} can be written as

$$\begin{aligned}&\left(\prod_{a=1}^{k-1}{}_\star e^{U_a} \star e^T\right) \star e^{U_k} \\ &\quad = \left(e^{(k-1)T+\sum_{a=1}^{k-1}U_a}\left(1 + \frac{i\epsilon}{4\pi}\sum_{a=1}^{k-1}\dot{T}U_a'(k-2a+1) + \frac{\epsilon^2}{32\pi^2}f(p,q) + \mathcal{O}(\epsilon^3)\right)\right) \star e^{U_k}\,,\end{aligned} \tag{2.192}$$

Where $f(p,q)$ is the ϵ^2 coefficient of the piece in parentheses in the first line. Expanding the remaining star product, the resulting ϵ^2 term is given by a shift of

$f(p,q)$

$$\frac{\epsilon^2}{32\pi^2}e^{(k-1)T+\sum_{a=1}^k U_a}\Bigg[f(p,q)+2\sum_{a=1}^{k-1}\big(\ddot{T}+(k-1)\dot{T}^2\big)U_a'U_k'(k-2a+1) \\ -\big((k-1)\ddot{T}+(k-1)^2\dot{T}^2\big)(U_k''+U_k'^2)\Bigg]. \tag{2.193}$$

Similarly the star product with the k^{th} e^T is

$$\Bigg(\Bigg(\prod_{a=1}^{k-1}{}_{\star} e^{U_a}\star e^T\Bigg)\star e^{U_k}\Bigg)\star e^T \\ =\Bigg(e^{(k-1)T+\sum_{a=1}^k U_a}\Bigg(1+\frac{i\epsilon}{4\pi}\sum_{a=1}^k \dot{T}U_a'(k-2a+1)+\frac{\epsilon^2}{32\pi^2}g(p,q)+\mathcal{O}(\epsilon^3)\Bigg)\Bigg)\star e^T\,, \tag{2.194}$$

and by expanding the remaining star product we get an ϵ^2 term

$$\frac{\epsilon^2}{32\pi^2}e^{kT+\sum_{a=1}^k U_a}\Bigg[g(p,q)-2\sum_{a=1}^k\dot{T}^2\Big(U_a''+U_a'\sum_{j=1}^k U_j'\Big)(k-2a+1) \\ -(\ddot{T}+\dot{T}^2)\Big(\sum_{a=1}^k U_a''+\Big(\sum_{a=1}^k U_a'\Big)^2\Big)\Bigg] \tag{2.195}$$

Using these expansions it is easy to build up the expansion up to ϵ^2 of the full density operator, by summing over the contributions from each star product

$$\rho_W=e^{LT+\sum_{a=1}^L U_a}\Bigg[1+\frac{i\epsilon}{4\pi}\sum_{a=1}^L\dot{T}U_a'(L-2a+2) \\ +\frac{\epsilon^2}{32\pi^2}\Bigg(2\sum_{k=2}^L\Bigg(\sum_{a=1}^{k-1}\big(\ddot{T}+(k-1)\dot{T}^2\big)U_a'U_k'(k-2a+1)-\big((k-1)\ddot{T}+(k-1)^2\dot{T}^2\big)(U_k''+U_k'^2)\Bigg) \\ -\sum_{k=1}^L\Bigg(2\sum_{a=1}^k\dot{T}^2\Big(U_a''+U_a'\sum_{j=1}^k U_j'\Big)(k-2a+1)+(\ddot{T}+\dot{T}^2)\Big(\sum_{a=1}^k U_a''+\Big(\sum_{a=1}^k U_a'\Big)^2\Big)\Bigg)+\mathcal{O}(\epsilon^3)\Bigg] \tag{2.196}$$

The coefficients of this expansion are $\rho_{(0)}$, $\rho_{(1)}$ and $\rho_{(2)}$, which should be plugged directly into (2.164). Integrating by parts to remove any double derivatives and plugging in (2.189), we find $Z_{l(0)}$, $Z_{l(1)}$ and $Z_{l(2)}$ are given by

$$Z_{l(0)}=\int dpdq\frac{1}{\operatorname{ch}^{l\nu}q\operatorname{ch}^{lL}p}\,,\qquad Z_{l(1)}=\frac{i\pi}{4}\int dpdq\frac{\operatorname{th}p\operatorname{th}q}{\operatorname{ch}^{l\nu}q\operatorname{ch}^{lL}p}l\sum_{a=1}^L n^{(a)}(L-2a+2)\,, \\ Z_{l(2)}=\frac{-\pi^2}{96}\int dpdq\frac{\operatorname{th}^2p\operatorname{th}^2q}{\operatorname{ch}^{l\nu}q\operatorname{ch}^{lL}p}l^2\big(\nu^2L^2(l^2-1)-12\Sigma\big)\,, \tag{2.197}$$

where

$$\nu = \sum_{a=1}^{L} n^{(a)}\,, \qquad \Sigma = \sum_{i>j} n^{(i)} n^{(j)} (i-j)(i-j-L)\,. \tag{2.198}$$

These integrals are easily carried out using (2.180), and we find

$$\begin{aligned} Z_{l(0)} &= \frac{1}{2^{l(\nu+L)}\pi} \frac{\Gamma(\frac{1}{2}\nu l)\Gamma(\frac{1}{2}Ll)}{\Gamma(\frac{1}{2}(\nu l+1))\Gamma(\frac{1}{2}(Ll+1))}\,, \qquad Z_{l(1)} = 0 \\ Z_{l(2)} &= \frac{-\pi}{3\cdot 2^{l(\nu+L)+7}} l^2\big(\nu^2 L^2 (l^2-1) - 12\Sigma\big) \frac{\Gamma(\frac{1}{2}\nu l)\Gamma(\frac{1}{2}Ll)}{\Gamma(\frac{1}{2}(\nu l+3))\Gamma(\frac{1}{2}(Ll+3))}\,. \end{aligned} \tag{2.199}$$

As we did for the $\hat{D}$ quivers, the next step is to compute the asymptotic expansion of the grand potential $J(\mu)$. Recall that the Fermi gas formulation of the $\hat{A}$ quivers (2.65) differs from that of the $\hat{D}$ quivers (2.81) by a factor of 2^{n^σ}, which led to the factor of $\frac{1}{2}$ in front of Z_l in (2.142). Without this factor (2.144) becomes

$$J_{(n)}(\mu) = -\sum_{l=1}^{\infty} \frac{(-1)^l Z_{l(n)} e^{\mu l}}{l} \tag{2.200}$$

Using a Mellin Barnes representation as in (2.183) to extract the asymptotic behaviour of $J(\mu)$, we again find an asymptotic expansion of the form (2.154), with C and B coefficients

$$C = \frac{2}{\pi^2 L\nu}, \qquad B = \frac{(L^2-4)(\nu^2-4) + 12\Sigma}{24L\nu}\,. \tag{2.201}$$

Again these coefficients do not receive contributions from higher order terms in ϵ, as was demonstrated for $\hat{A}$ quivers already in [14].

Chapter 3

Superconformal indices of 4d theories

In this chapter we study superconformal indices of 4d theories with $\mathcal{N} \geqslant 2$ supersymmetry, which are closely related to the 3d theories we considered in section 2.1. These theories are analagously constructed from $\mathcal{N} = 2$ vector and hypermultiplets, which are directly related through dimensional reduction to the 3d $\mathcal{N} = 4$ hypermultiplets of section 2.1.3.

We begin by reviewing the definition of the superconformal index and the Schur limit, before reviewing in section 3.2 how the index may be reduced to an elliptic matrix model, by a careful counting of the states that can contribute. In the remaining sections we turn to the analysis of these matrix models using a Fermi gas approach, presenting work we did in [3, 4].

3.1 The $\mathcal{N} = 2$ superconformal index and the Schur limit

As mentioned in the introduction, the superconformal index, introduced in [57, 58], is a generalisation of the Witten index [56] that counts states according to their quantum numbers and their fermionic/bosonic nature, through the trace formula

$$\mathcal{I} = \mathrm{Tr}(-1)^F e^{-\beta\{Q,Q^\dagger\}} \prod_i \kappa_i^{G_i} \,. \tag{3.1}$$

Here F is the fermion number, Q, $Q^\dagger$ are a pair of conjugate supercharges, and κ_i are fugacities for combinations of superconformal or flavour charges that commute with both Q and $Q^\dagger$. Following the usual Witten argument [56], states which are not annihilated by $\{Q, Q^\dagger\}$ necessarily come in bosonic/fermionic pairs, whose contributions to (3.1) then cancel exactly due to the $(-1)^F$. As a result the index is independent of β.

We are interested specifically in superconformal indices for theories on $S^3 \times S^1$ with (at least) $\mathcal{N} = 2$ supersymmetry. The $\mathcal{N} = 2$ superconformal algebra on $S^3 \times S^1$ is $SU(2,2|2)$, and states are labelled by the corresponding quantum numbers (E, j_1, j_2, R, r), where E is the energy, (j_1, j_2) are the Cartans of the $SU(2)_1 \times SU(2)_2$ isometries of S^3 and (R, r) are the Cartans of the R symmetry $SU(2)_R \times U(1)_r$. A particular choice of supercharges in (3.1) has the anticommutation relation

$$2\{Q, Q^\dagger\} = E - 2j_2 - 2R + r\,. \tag{3.2}$$

The most general index corresponding to this choice of supercharges is then

$$\mathcal{I} = \mathrm{Tr}_{\mathcal{H}}\,(-1)^F p^{E+2j_1-2R-r} q^{E-2j_1-2R-r} t^{2R+2r} \prod_a e^{2iu^{(a)}F^{(a)}}\,, \tag{3.3}$$

where $\mathrm{Tr}_{\mathcal{H}}$ denotes a trace over states satisfying $E - 2j_2 - 2R + r = 0$ and p, q and t are fugacities for the three linear combinations of superconformal charges that commute with Q and $Q^\dagger$, while $e^{2iu^{(a)}}$ are fugacities for flavour charges $F^{(a)}$.

The Schur limit of the index is given by imposing $t = q$, giving

$$\mathcal{I}_{\mathrm{schur}} = \mathrm{Tr}_{\mathcal{H}}\,(-1)^F p^{E+2j_1-2R-r} q^{E-2j_1+r} \prod_a e^{2iu^{(a)}F^{(a)}}\,. \tag{3.4}$$

In fact, the linear combination of superconformal charges coupling to the fugacity p is given by the anticommutator of a different pair of supercharges

$$2\{Q' Q'^\dagger\} = E + 2j_1 - 2R - r\,. \tag{3.5}$$

Furthermore, the combination of supercharges coupling to the fugacity q happens to commute with both Q' and $Q'^\dagger$. Following the Witten argument, this immediately implies that the Schur index does not depend on the fugacity p and counts states annihilated by both of the anticommutators (3.2) and (3.5). Then (3.4) can be written as simply

$$\mathcal{I}_{\mathrm{schur}} = \mathrm{Tr}_{\mathcal{H}'}\,(-1)^F q^{2(E-R)} \prod_a e^{2iu^{(a)}F^{(a)}}\,, \tag{3.6}$$

where $\mathrm{Tr}_{\mathcal{H}'}$ now denotes a trace over states satisfying both $E - 2j_2 - 2R + r = 0$ and $E + 2j_1 - 2R - r = 0$.

3.2 Evaluating the index as a matrix model

Here we review how the $\mathcal{N} = 2$ Schur index (3.6) may be evaluated as a matrix model [57, 59], by a systematic counting of states that contribute to the index.

The first step is to count contributions from only single particle states

$$\mathcal{D}\,\phi\,|0\rangle \tag{3.7}$$

where ϕ can be any component field (or 'letter') appearing in one of the multiplets of the theory, and $\mathcal{D}$ is any combination of spacetime derivatives .

The $\mathcal{N} = 2$ vector multiplets have component fields $(\phi, \bar{\phi}, \lambda^I_\alpha \bar{\lambda}^I_{\dot{\alpha}}, F_{\alpha\beta}, \bar{F}_{\dot{\alpha},\dot{\beta}})$ (where $\alpha = \pm$, $\dot{\alpha} = \dot{\pm}$ are Lorentz indices corresponding to $SU(2)_1$ and $SU(2)_2$ respectively, and $I = 1, 2$ are $SU(2)_R$ R symmetry indices) while the $\mathcal{N} = 2$ hypermultiplet is built out of two conjugate $\mathcal{N} = 1$ chiral multiplets which have component fields $(q, \bar{q}, \psi_\alpha, \bar{\psi}_{\dot{\alpha}})$ and $(\tilde{q}, \bar{\tilde{q}}, \tilde{\psi}_\alpha, \bar{\tilde{\psi}}_{\dot{\alpha}})$, and whose fields carry opposite $U(1)$ flavour charges. Since we are interested in the Schur index, we only need to concern ourselves with letters and spacetime derivatives whose superconformal charges satisfy that both (3.2) and (3.5) vanish. The list is rather short; the relevant letters and space time derivatives along with their superconformal charges are tabulated below

letter	$\{E,\, j_1,\, j_2,\, R,\, r\}$	$2(E-R)$
λ^1_-	$\{\frac{3}{2}, -\frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}\}$	2
$\bar{\lambda}^1_{\dot{+}}$	$\{\frac{3}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$	2
q	$\{1, 0, 0, \frac{1}{2}, 0\}$	1
$\tilde{q}$	$\{1, 0, 0, \frac{1}{2}, 0\}$	1
$\partial_{-\dot{+}}$	$\{1, -\frac{1}{2}, \frac{1}{2}, 0, 0\}$	2

With these data one can easily evaluate the single letter indices, that count all states of the form (3.7). Indeed, for the fields of an (ungauged) vector multiplet the contribution to the single letter index is

$$i^{\text{vec}}(q, e^{2iu}) = \sum_{n=0}^{\infty} q^{2n}(-q^2 - q^2) = \frac{-2q^2}{1-q^2}\,, \tag{3.8}$$

where the sum comes from allowing any number of $\partial_{-\dot{+}}$ derivatives.

Similarly the total contribution from an (ungauged) hypermultiplet is

$$i^{\text{hyp}}(q, e^{2iu}) = \sum_{n=0}^{\infty} q^{2n}(qe^{2iu} + qe^{-2iu}) = \frac{q}{1-q^2}(e^{2iu} + e^{-2iu})\,. \tag{3.9}$$

The full index should count contributions from all possible 'words' that can be built out of letters. For example the state

$$\lambda_-^1 q \left|0\right\rangle \tag{3.10}$$

would give a contribution $-q^3e^{2iu}$ to the Schur index.

If we allow the multiplets to become gauged under a gauge group U, then a word such as (3.10) will transform in some product representation of the gauge group. Since the index should count only gauge invariant states, we need to determine the number of singlet factors in this product representation. Such a counting can be achieved by exploiting the orthogonality of group characters, which gives the formula [88]

$$n_{\text{singlets}} = \int [dU] \chi_{R_1\otimes R_2 \otimes \cdots}(U)\,, \tag{3.11}$$

where $[dU]$ is the Harr measure, and $\chi_{R_1\otimes R_2\otimes\cdots}(U)$ is the character of the group element U in the product representation $R_1 \otimes R_2 \otimes \cdots$. The characters of product representations also obey a helpful factorisation property

$$\chi_{R_1\otimes R_2}(U) = \chi_{R_1}(U)\cdot\chi_{R_2}(U)\,. \tag{3.12}$$

Returning then to the example of (3.10), if λ_-^1 and q transform respectively under representations R_1 and R_2 of the gauge group, then this word should give a contribution to the Schur index of

$$-q^3e^{2iu}\int [dU]\,\chi_{R_1}(U)\chi_{R_2}(U)\,. \tag{3.13}$$

To evaluate the Schur index exactly we now need to sum the contributions from all possible words. In fact, one can capture all of these contributions by taking a plethystic exponentiation [89] of the single letter index, which gives the formula

$$\mathcal{I} = \int [dU] \prod_a \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} i^a(q^n, e^{2inu^{(a)}})\chi_{R_a}(U^n)\right]\,, \tag{3.14}$$

where i^a denotes the single letter index of the a^{th} multiplet, whose fields transform in a representation R_a of the gauge group.

Before simplifying this matrix model further we pause to collect some group theory data that will be important for the particular gauge theories we will study.

For unitary $U(N)$ gauge groups the (gauge fixed)[1] harr measure can be written as

$$\int [dU] = \frac{1}{\pi^N N!}\int_0^{\pi} \prod_{i=1}^{N} d\alpha_i \prod_{i<j} 4\sin^2 2(\alpha_i - \alpha_j)\,, \tag{3.15}$$

[1]See the discussion on gauge fixing after (2.27)

where the integration is over the N eigenvalues $e^{2i\alpha_i}$ parameterising the Cartan of $U(N)$. For $SU(N)$ gauge groups one simply needs to include in the measure an additional periodic delta function constraint $\delta(\sum_{i=1}^{N} \alpha_i)$.

The characters of adjoint representations for $U(N)$ and $SU(N)$ are

$$\chi_{\text{adjoint}\,U(N)} = \sum_{i,j} e^{2i(\alpha_i - \alpha_j)} \qquad \chi_{\text{adjoint}\,SU(N)} = \sum_{i,j} e^{2i(\alpha_i - \alpha_j)} - 1 \tag{3.16}$$

The characters for bifundamental representations are the same whether they transform under $U(N) \times U(N)$ or $SU(N) \times SU(N)$. Distinguishing the eigenvalues of the two group factors by giving one set a prime, the characters are

$$\chi_{\text{bifundamental}} = \sum_{i,j} e^{2i(\alpha_i - \alpha'_j)}. \tag{3.17}$$

Putting these ingredients together, the contribution to the integrand of (3.14) from $\mathcal{N} = 2$ vector and hypermultiplets can then be written elegantly in terms of Jacobi theta functions and Dedekind eta functions, whose definitions can be found in appendix E.

For example, the contribution of a $U(N)$ vector multiplet in the adjoint representation can be written as

$$\begin{aligned} \prod_{i,j} \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{-2q^{2n}}{1-q^{2n}} e^{2in(\alpha_i - \alpha_j)}\right] &= \prod_{i,j} \exp\left[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{-2q^{2nk}}{n} e^{2in(\alpha_i - \alpha_j)}\right] \\ &= \prod_{i,j}\prod_{k=1}^{\infty} \left(1 - q^{2k} e^{2i(\alpha_i - \alpha_j)}\right)^2 \\ &= \left(\eta(\tau) q^{-\frac{1}{12}}\right)^{2N} \prod_{i<j} \frac{\vartheta_1^2(\alpha_i - \alpha_j, q)}{4\sin^2 2(\alpha_i - \alpha_j) q^{\frac{1}{3}} \eta^2(\tau)}\,, \end{aligned} \tag{3.18}$$

where τ is the quasi period related to the nome q through $q = e^{i\pi\tau}$. The trigonometric factor in the denominator exactly cancels against the Vandermonde determinant coming from (3.15). If the gauge group is $SU(N)$ rather than $U(N)$, then the extra -1 in the adjoint character (3.16) will simply contribute an additional factor $\left(\eta(\tau) q^{-\frac{1}{12}}\right)^{-2}$

Similarly a hypermultiplet transforming in the adjoint representation of $U(N)$ gives a matrix model contribution

$$\prod_{i,j} \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{2q^{n}}{1-q^{2n}} e^{2in(\alpha_i - \alpha_j + u)}\right] = \prod_{i,j} \frac{q^{-\frac{1}{12}} \eta(\tau)}{\vartheta_4(\alpha_i - \alpha_j + u, q)}\,. \tag{3.19}$$

Again due to the -1 in the character for the $SU(N)$ adjoint representation (3.16), the contribution from an $SU(N)$ adjoint hypermultiplet is modified by a factor of

$q^{\frac{1}{12}}\vartheta_4(u,q)\eta^{-1}(\tau)$

Finally a bifundamental hypermultiplet of $U(N)\times U(N)$ or $SU(N)\times SU(N)$ contributes a factor

$$\prod_{i,j}\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\frac{2q^n}{1-q^{2n}}e^{2in(\alpha_i-\alpha'_j+u)}\right]=\prod_{i,j}\frac{q^{-\frac{1}{12}}\eta(\tau)}{\vartheta_4(\alpha_i-\alpha'_j+u,q)}\,. \tag{3.20}$$

For $U(N)$ vector multiplets there is the possibility of adding FI terms. As discussed in [90], on $S^3\times S^1$ the FI parameter ζ is quantised and must be an integer. The contribution to the matrix model turns out to be a factor

$$\prod_{i=1}^{N}e^{2\pi i\zeta\alpha_i}\,. \tag{3.21}$$

The ingredients (3.18), (3.19), (3.20) and (3.21) together with the measure (3.15), allow one to very quickly associate a matrix model to the Schur index of any quiver theory involving $U(N)$ or $SU(N)$ nodes and bifundamental or adjoint matter.

3.3 From matrix models to free fermions

Here we discuss how the matrix models corresponding to Schur indices of 4d $\mathcal{N}=2$ theories can be written in terms of partition functions of free fermions. The theories we discuss are $\hat{A}_{L-1}$ quiver theories with gauge group $U(N)^L$ or $SU(N)^L$. There are $\mathcal{N}=2$ vector multiplets for each gauge group factor and $\mathcal{N}=2$ bifundamental hypermultiplets connecting them in circular fashion. Much as was the case for partition functions of 3d $\mathcal{N}\geqslant 3$ $\hat{A}$ theories (see section 2.4.1), the analysis hinges crucially on a determinant identity that can resolve the matrix model integrand into a series of kernels between successive nodes. In this case we need an elliptic generalisation of the Cauchy determinant identity used in the 3d case, which we describe in appendix F.

There are further striking differences with the 3d calculation: after factorising the matrix model into a series of determinants we are left (apart from in the special case of $\mathcal{N}=4$ SYM) with centre of mass terms that complicate the calculation and impede an immediate free fermion interpretation. To get around this, we expand these terms as Fourier series, which resolves the centre of mass interactions at the cost of introducing an infinite number of terms, each of which can be interpreted as a free Fermi gas partition function. We proceed to study these Fermi gasses individually by moving to the grand ensemble and considering their grand partition functions, which we find to take a particularly elegant form as a product of Jacobi theta functions evaluated at roots of a certain polynomial. From this expression we are able to extract large N asymptotic results for the index, as well as exact, all order in N results for shorter quivers, which we will come to in sections 3.5 and 3.6.

3.3.1 Circular quivers as free Fermi gases

Using the formulae given in section 3.2, the matrix model for the Schur index of a $U(N)$ circular quiver gauge theory with L nodes is

$$\begin{aligned}\mathcal{I} = &\frac{q^{-\frac{LN^2}{4}}\eta(\tau)^{3NL}}{N!^L\pi^{NL}} \\ &\times \int_0^\pi \prod_{a=1}^L d^N\alpha^{(a)} e^{2i\zeta^{(a)}\sum_{i=1}^N \alpha_i^{(a)}} \frac{\prod_{i<j}\vartheta_1\big(\alpha_i^{(a)}-\alpha_j^{(a)}\big)\vartheta_1\big(\alpha_i^{(a+1)}-\alpha_j^{(a+1)}\big)}{\prod_{i,j}\vartheta_4\big(\alpha_i^{(a)}-\alpha_j^{(a+1)}+u^{(a)}\big)}\,,\end{aligned} \tag{3.22}$$

where $\alpha^{(a)}$ are the eigenvalues of the a'th $U(N)$ node and it is understood that $\alpha_i^{(L+1)} = \alpha_i^{(1)}$, and we have used the notation $\vartheta_i(z) = \vartheta_i(z,q)$. We allow here for arbitrary FI parameters $\zeta^{(a)}$ for the gauge fields and arbitrary chemical potentials $u^{(a)}$ for the flavour symmetries of the hypermultiplets.

The integrand in (3.22) can be rewritten using an elliptic determinant identity (F.4) given in appendix F

$$\begin{aligned}&\frac{\prod_{i<j}\vartheta_1\big(\alpha_i^{(a)}-\alpha_j^{(a)}\big)\vartheta_1\big(\alpha_i^{(a+1)}-\alpha_j^{(a+1)}\big)}{\prod_{i,j}\vartheta_4\big(\alpha_i^{(a)}-\alpha_j^{(a+1)}+u^{(a)}\big)} = \left(\frac{\vartheta_2}{\vartheta_4\vartheta_3}\right)^N \\ &\quad\times \frac{\vartheta_3\, q^{-\frac{N^2}{4}} e^{iN^2(A^{(a)}-A^{(a+1)}+u^{(a)})}}{\vartheta_3\big(N(A^{(a)}-A^{(a+1)}+u^{(a)}+\frac{\pi\tau}{2})\big)} \det_{ij}\left(\mathrm{cn}\left(\big(\alpha_i^{(a)}-\alpha_j^{(a+1)}+u^{(a)}\big)\vartheta_3^2\right)\right).\end{aligned} \tag{3.23}$$

Here we have used the notation $\vartheta_i = \vartheta_i(0,q)$ and introduced the centres of mass

$$A^{(a)} = \frac{1}{N}\sum_{i=1}^N \alpha_i^{(a)}\,. \tag{3.24}$$

Putting this together allows us to write the index as

$$\begin{aligned}\mathcal{I} = \frac{q^{-\frac{N^2L}{2}} q^{iN^2\sum_a u^{(a)}}}{N!^L} \int_0^\pi \prod_{a=1}^L &\left(d^N\alpha^{(a)} \frac{\vartheta_3}{\vartheta_3\big(N(A^{(a)}-A^{(a+1)}+u^{(a)}+\frac{\pi\tau}{2})\big)}\right) \\ &\times \det_{ij}\left(e^{2i\zeta^{(a)}\alpha_i^{(a)}} \frac{\vartheta_2^2}{2\pi}\mathrm{cn}\left(\big(\alpha_i^{(a)}-\alpha_j^{(a+1)}+u^{(a)}\big)\vartheta_3^2\right)\right),\end{aligned} \tag{3.25}$$

where we have used the identity (E.7) to simplify the α-independent factor.[2]

Each determinant can then be written as a sum over permutations, and by relabelling the eigenvalues, one can factor all but one of the permutations, picking

[2]The factor of $\vartheta_2^2/2\pi$ is included with the cn functions to simplify their Fourier expansion below.

up a factor of $N!^{L-1}$ and leading to

$$\mathcal{I} = \frac{q^{-N^2(\frac{L}{2}-iU)}}{N!} \sum_{\sigma\in S^N} (-1)^\sigma \int_0^\pi \prod_{a=1}^{L} \Bigg(d^N\alpha^{(a)} \frac{\vartheta_3}{\vartheta_3\big(N(A^{(a)}-A^{(a+1)}+u^{(a)}+\frac{\pi\tau}{2})\big)}\Bigg)$$
$$\times \prod_{i=1}^{N} \Bigg(\prod_{a=1}^{L-1} e^{2i\zeta^{(a)}\alpha_i^{(a)}} \frac{\vartheta_2^2}{2\pi} \operatorname{cn}\big((\alpha_i^{(a)}-\alpha_i^{(a+1)}+u^{(a)})\vartheta_3^2\big) \tag{3.26}$$
$$\times\, e^{2i\zeta^{(L)}\alpha_i^{(L)}} \frac{\vartheta_2^2}{2\pi} \operatorname{cn}\big((\alpha_i^{(L)}-\alpha_{\sigma(i)}^{(1)}+u^{(L)})\vartheta_3^2\big)\Bigg), \qquad U=\sum_{a=1}^{L} u^{(a)}\,.$$

This expression strongly suggests that the eigenvalues describe fermionic degrees of freedom. The difficulty in writing down a single particle density operator comes from the presence of the centre of mass coordinates $A^{(a)}$ which introduce complicated interactions. It is possible to overcome this problem by expanding the centre of mass dependent terms in Fourier modes, generating a weighted sum over partition functions of *free* Fermi gasses. We explain this now separately for the cases of $SU(N)$ and $U(N)$ gauge group factors.

3.3.2 $SU(N)$ quivers

In the case of a quiver with $SU(N)$ nodes, the centre of mass parameters $A^{(a)}$ all vanish. The elimination of this degree of freedom has four important consequences: First, it simplifies the denominator in the first line of (3.26). Second, there is an extra overall factor of $q^{\frac{L}{6}}\eta(\tau)^{-2L}$, as discussed after (3.18).[3] Third, there cannot be any FI parameters. Lastly, the eigenvalues $\alpha_i^{(a)}$ are subject to the (periodic) delta function constraint $\delta\big(NA^{(a)}\big)$. We choose to represent all but one of these L delta function constraints in difference form $\delta\big(NA^{(L)}\big)\prod_{a=1}^{L-1}\delta\big(N(A^{(a)}-A^{(a+1)})\big)$.

From these considerations, (3.26) becomes

$$\mathcal{I} = \frac{q^{-N^2(\frac{L}{2}-iU)}q^{\frac{L}{6}}}{\eta^{2L}(\tau)} \prod_{a=1}^{L} \frac{\vartheta_3}{\vartheta_3\big(Nu^{(a)}+N\frac{\pi\tau}{2}\big)}\, Z(N) \tag{3.27}$$

[3]For the single node theory ($\mathcal{N}=4$ SYM) the hypermultiplet, is in the adjoint rather then bifundamental representation which for $SU(N)$ groups introduces an additional factor of $q^{\frac{1}{12}}\eta^{-1}(\tau)\vartheta_4(u)$, as discussed after (3.19)

where $Z(N)$ is a rescaled index, given by

$$\begin{aligned}Z(N) = \sum_{\sigma\in S^N}(-1)^\sigma &\int_0^\pi \prod_{a=1}^{L-1} d^N\alpha^{(a)}\delta\big(N(A^{(a)}-A^{(a+1)}-u^{(a)})\big)\\ &\times\int_0^\pi d^N\alpha^{(L)}\delta\big(N(A^{(L)}+U-u^{(L)})\big)\\ &\times\prod_{i=1}^N\left(\frac{\vartheta_2^2}{2\pi}\operatorname{cn}\big((\alpha_i^{(L)}-\alpha_{\sigma(i)}^{(1)}+U)\vartheta_3^2\big)\prod_{a=1}^{L-1}\frac{\vartheta_2^2}{2\pi}\operatorname{cn}\big((\alpha_i^{(a)}-\alpha_i^{(a+1)})\vartheta_3^2\big)\right),\end{aligned} \tag{3.28}$$

where we have shifted the eigenvalues as $\alpha^{(a)} \to \alpha^{(a)} + \sum_{b=1}^{a-1} u^{(b)}$ so that the u's appear only inside the delta functions and one cn.

Though there is no longer any dependence on the centre of masses $A^{(a)}$ in the theta functions in (3.27), they still appear in the delta functions. To remedy that we represent the delta functions by their Fourier expansion

$$\begin{aligned}&\delta\big(N(A^{(L)}+U-u^{(L)})\big)\prod_{a=1}^{L-1}\delta\big(N(A^{(a)}-A^{(a+1)}-u^{(a)})\big)\\ &=\sum_{\vec{n}\in\mathbb{Z}^L} e^{-2iN\sum_{a=1}^L n^{(a)}u^{(a)}}e^{2iNUn^{(L)}}e^{2in^{(L)}\sum_i\alpha_i^{(L)}}\prod_{a=1}^{L-1}e^{2in^{(a)}\sum_i(\alpha_i^{(a)}-\alpha_i^{(a+1)})},\end{aligned} \tag{3.29}$$

where $\vec{n} = \{n^{(a)}\}$. We can then write the rescaled index as a sum

$$Z(N) = \sum_{\vec{n}\in\mathbb{Z}^L} e^{-2iN\sum_{a=1}^L n^{(a)}u^{(a)}}e^{2iNUn^{(L)}}Z_{\vec{n}}\,. \tag{3.30}$$

Now each Fourier coefficient $Z_{\vec{n}}$ is a partition function of a *free* Fermi gas, whose expression in terms of a single particle density operator ρ is identical (up to the range of integration) to (2.65)

$$Z_{\vec{n}}(N) = \frac{1}{N!}\sum_{\sigma\in S^N}(-1)^\sigma\int_0^\pi d^N\alpha_i\prod_i^N\rho_{\vec{n}}(\alpha_i,\alpha_{\sigma(i)})\,. \tag{3.31}$$

The density operator in this case is

$$\begin{aligned}\rho_{\vec{n}}\big(\alpha^{(1)},\alpha^{(L+1)}\big) = &\int_0^\pi\prod_{a=2}^L d\alpha^{(a)}\,e^{2in^{(L)}\alpha^{(L)}}\frac{\vartheta_2^2}{2\pi}\prod_{a=1}^{L-1}e^{2in^{(a)}(\alpha^{(a)}-\alpha^{(a+1)})}\\ &\times\operatorname{cn}\big((\alpha^{(L)}-\alpha^{(L+1)}+U)\vartheta_3^2\big)\prod_{a=1}^{L-1}\frac{\vartheta_2^2}{2\pi}\operatorname{cn}\big((\alpha^{(a)}-\alpha^{(a+1)})\vartheta_3^2\big)\,.\end{aligned} \tag{3.32}$$

Note that the Fourier modes $n^{(a)}$ play a role in $Z_{\vec{n}}$ analogous to the FI parameters $\zeta^{(a)}$ in (3.26) and couple to the flavour chemical potentials $u^{(a)}$ in the expansion

(3.30).

As we did when we analysed Fermi gasses coming from 3d partition functions (see section 2.6.1), we proceed by decomposing (3.31) into spectral traces $Z_{\vec{n};\ell}$

$$Z_{\vec{n};\ell} = \mathrm{Tr}(\rho_{\vec{n}}^{\ell}) = \int_0^{\pi} dx_1 \cdots dx_\ell \, \rho_{\vec{n}}(x_1, x_2) \cdots \rho_{\vec{n}}(x_\ell, x_1) \, . \tag{3.33}$$

Indeed, conjugacy classes of S_N have m_ℓ cycles of length ℓ, and from the definition (3.31) of $Z_{\vec{n}}(N)$ we get (*c.f.* (2.142))

$$Z_{\vec{n}}(N) = {\sum_{\{m_\ell\}}}' \prod_\ell \frac{Z_{\vec{n};\ell}^{m_\ell} \, (-1)^{(\ell-1)m_\ell}}{m_\ell! \, \ell^{m_\ell}} \, , \tag{3.34}$$

where the prime denotes a sum over sets that satisfy $\sum_\ell \ell m_\ell = N$.

To evaluate $Z_{\vec{n};\ell}$, we first simplify the expression for the density $\rho_{\vec{n}}$ (3.32) by using the Fourier expansion of the elliptic function

$$\mathrm{cn}\left(z \, \vartheta_3^2\right) = \frac{1}{\vartheta_2^2} \sum_{p \in \mathbb{Z}} \frac{e^{i(2p-1)z}}{\cosh i\pi\tau(p - \frac{1}{2})} \, , \tag{3.35}$$

and we obtain

$$\begin{aligned} \rho_{\vec{n}} = & \sum_{\vec{p} \in \mathbb{Z}^L} e^{i(2p^{(L)}-1)U} \prod_{a=1}^{L} \frac{1}{2\pi} \frac{1}{\cosh i\pi\tau(p^{(a)} - \frac{1}{2})} \\ & \times \int_0^{\pi} \prod_{a=2}^{L} d\alpha^{(a)} \, e^{2in^{(L)}\alpha^{(L)}} e^{2i(p^{(L)}-\frac{1}{2})(\alpha^{(L)}-\alpha^{(L+1)})} \prod_{a=1}^{L-1} e^{2i(n^{(a)}+p^{(a)}-\frac{1}{2})(\alpha^{(a)}-\alpha^{(a+1)})} . \end{aligned} \tag{3.36}$$

Shifting the summation over $p^{(a)} \to p^{(a)} - n^{(a)}$, and doing the integration over the $\alpha^{(a)}$'s gives

$$\rho_{\vec{n}} = \frac{1}{\pi} \sum_{p \in \mathbb{Z}} e^{2i(p-n^{(L)}-\frac{1}{2})U} e^{2i(p-\frac{1}{2})\alpha^{(1)}} e^{-2i(p-n^{(L)}-\frac{1}{2})\alpha^{(L+1)}} \prod_{a=1}^{L} \frac{1}{2\cosh i\pi\tau\left(p - n^{(a)} - \frac{1}{2}\right)} \, . \tag{3.37}$$

As explained above, we are interested in computing the quantity $Z_{\vec{n};\ell}$ (3.33). For $\ell = 1$ we find

$$Z_{\vec{n};1} = \int_0^{\pi} d\alpha \, \rho_{\vec{n}}(\alpha, \alpha) = \delta_{n^{(L)}} \sum_{p \in \mathbb{Z}} e^{2i(p-n^{(L)}-\frac{1}{2})U} \prod_{a=1}^{L} \frac{1}{2\cosh i\pi\tau\left(p - n^{(a)} - \frac{1}{2}\right)} \, . \tag{3.38}$$

This structure persists also when considering the convolution of several ρ's, with a constraint on $n^{(L)}$ and a single sum over p

$$Z_{\vec{n};\ell} = \delta_{n^{(L)}} \sum_{p\in\mathbb{Z}} e^{2i(p-n^{(L)}-\frac{1}{2})U\ell} \prod_{a=1}^{L} \left(\frac{1}{2\cosh i\pi\tau\big(p-n^{(a)}-\frac{1}{2}\big)} \right)^{\ell} . \tag{3.39}$$

The presence of the $\delta_{n^{(L)}}$ factor in the expression above tells us that the sum in (3.30) is in reality only over $\{n^{(a)}\} \in \mathbb{Z}^{L-1}$ with $n^{(L)} = 0$. From now on we omit this Kronecker delta, and the modes $n^{(a)}$ run over $a = 1, \cdots, L-1$.

We can plug the expressions (3.39) into (3.34) to evaluate $Z_{\vec{n}}(N)$ and then sum over the integers $\vec{n} \in \mathbb{Z}^{L-1}$ to find the index $\mathcal{I}_{SU(N)}$ (3.27), (3.30). Instead, we will consider the grand cannonical partition function associated with each $\vec{n} \in \mathbb{Z}^{L-1}$ (*c.f.* (2.143))

$$\Xi_{\vec{n}}(\kappa) = 1 + \sum_{N=1}^{\infty} Z_{\vec{n}}(N)\kappa^N , \qquad \kappa = e^{\mu} , \tag{3.40}$$

where κ is the fugacity and μ is the chemical potential. This definition is easily inverted to recover $Z_{\vec{n}}(N)$

$$Z_{\vec{n}}(N) = \frac{1}{2\pi i} \int_{-i\pi}^{i\pi} d\mu\, \Xi_{\vec{n}}(e^{\mu}) e^{-\mu N} . \tag{3.41}$$

The combinatorics simplify when considering the grand potential

$$J_{\vec{n}}(\mu) \equiv \log \Xi_{\vec{n}}\big(e^{\mu}\big) = -\sum_{\ell=1}^{\infty} \frac{(-1)^{\ell} Z_{\vec{n};\ell} e^{\mu\ell}}{\ell} , \tag{3.42}$$

and we can then easily sum over ℓ and find a very compact expression

$$\Xi_{\vec{n}}(\kappa) = \prod_{p\in\mathbb{Z}} \left(1 + \kappa e^{2i(p-n^{(L)}-\frac{1}{2})U} \prod_{a=1}^{L} \frac{1}{2\cosh i\pi\tau\big(p-n^{(a)}-\frac{1}{2}\big)} \right) . \tag{3.43}$$

From now on we focus on the case with $U = 0$, *i.e.* the product of the flavour fugacities is 1. This allows us to write $\Xi_{\vec{n}}$ as a product of theta functions evaluated at the roots of a polynomial. Indeed, for $U = 0$, each term in the product over p in (3.43) can be written as

$$\frac{X^L q^{-n}\,\kappa + \prod_{a=1}^{L}\big(1 + X^2 q^{-2n^{(a)}}\big)}{\prod_{a=1}^{L}\big(1 + X^2 q^{-2n^{(a)}}\big)} , \qquad X \equiv q^{p-\frac{1}{2}} , \tag{3.44}$$

where $n = \sum_{a=1}^{L} n^{(a)}$. The numerator of (3.44) is a polynomial of degree $2L$ in X

with coefficients that depend on q, $n^{(a)}$ and κ, but not on p. It can be factored as

$$\frac{\prod_{j=1}^{2L}\left(1+e^{2id_j}X\right)}{\prod_{a=1}^{L}\left(1+X^2q^{-2n^{(a)}}\right)}\,. \tag{3.45}$$

Now take the term in (3.43) with $p\to -p+1$. We can write it also as (3.44) with the same denominator. The numerator is then of a similar form with $n^{(a)}\to -n^{(a)}$, which is factorised by the inverse roots $-e^{2id_j}$. Splitting then the product in (3.43) over only positive p gives

$$\begin{aligned}\Xi_{\vec{n}} &= \prod_{p=1}^{\infty}\frac{\prod_{j=1}^{2L}\left(1+e^{2id_j}q^{p-\frac{1}{2}}\right)\left(1+e^{-2id_j}q^{p-\frac{1}{2}}\right)}{\prod_{a=1}^{L}\left(1+q^{-2n^{(a)}}q^{2p-1}\right)\left(1+q^{2n^{(a)}}q^{2p-1}\right)} = \frac{\prod_{j=1}^{2L}\vartheta_3\left(d_j,q^{\frac{1}{2}}\right)}{\vartheta_4^L\prod_{a=1}^{L}\vartheta_3(n^{(a)}\pi\tau,q)} \\ &= \frac{q^{\sum_{a=1}^{L}(n^{(a)})^2}}{\vartheta_4^L\vartheta_3^L}\prod_{j=1}^{2L}\vartheta_3\left(d_j,q^{\frac{1}{2}}\right).\end{aligned} \tag{3.46}$$

For even L the numerator of (3.44) can also be viewed as a polynomial of degree L in X^2.

Explicitly, equation (3.45) factorised into L terms is

$$\frac{\prod_{j=1}^{L}\left(1+e^{2i\tilde{d}_j}X^2\right)}{\prod_{a=1}^{L}\left(1+X^2q^{-2n^{(a)}}\right)}\,, \tag{3.47}$$

and the grand partition function is now expressed in terms of theta functions with nome q rather than $q^{\frac{1}{2}}$

$$\Xi_{\vec{n}}^{\text{even }L} = \frac{q^{\sum_{a=1}^{L}(n^{(a)})^2}}{\vartheta_3^L}\prod_{j=1}^{L}\vartheta_3(\tilde{d}_j)\,. \tag{3.48}$$

Clearly the $2L$ roots for X are given in terms of the new ones by the pairs $\pm e^{-i\tilde{d}_j}$ and the expressions (3.46) and (3.48) are related by a simple application of Watson's identity (E.9).

It is rather intriguing that the grand canonical partition function ends up also as a product of Jacobi theta functions, similar to the superconformal indices of the free hypermultiplets and vector multiplets. The reason for this is not clear to us, but it is a manifestation of the modular properties of the Schur index, discussed in [91]. The same can be said for the expressions we find for finite N in Section 3.6.

In the rest of this section we briefly comment on the case of $U(N)$ gauge groups. In Section 3.4 we compute the d_j's at leading order in the large μ expansion, from which we obtain the leading large N contribution to the index. In Section 3.5 we focus on the cases of $L=1$ and $L=2$, for which the numerator of (3.44) is quadratic and so can be easily factored algebraically, and the roots obtained exactly.[4] This allows us to go beyond the large N limit, and obtain an exact all order expression for the index.

3.3.3 $U(N)$ quivers

A slight modification of the approach above allows to study quiver theories with $U(N)$ nodes. In that case the centre of mass dependence is not in a delta function but in the inverse Jacobi theta functions in the first line of (3.26). Those too can be expanded in a Fourier series.

From the expression

$$\frac{\vartheta_3}{\vartheta_3(z)} = \frac{2q^{\frac{1}{4}}}{\vartheta_2\vartheta_4}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{n(n+1)}}{1+e^{2iz}q^{2n+1}}\,, \tag{3.49}$$

we obtain the Fourier coefficients

$$\frac{\vartheta_3}{\vartheta_3(z)} = \sum_{n\in\mathbb{Z}} F_n e^{2inz}\,, \qquad F_n = \frac{2q^{\frac{1}{4}}}{\vartheta_2\vartheta_4}\sum_{k=0}^{\infty}(-1)^k q^{k(k+1)}(-1)^n q^{|n|(2k+1)}\,. \tag{3.50}$$

The index (3.26) then becomes

$$\mathcal{I}_{U(N)} = q^{-\frac{N^2L}{2}} q^{iN^2U} e^{2i\sum_{a=1}^{L}(\zeta^{(a)}\sum_{b=1}^{a-1}u^{(b)})}\sum_{\vec{n}\in\mathbb{Z}^L}\left(e^{2in^{(L)}NU}\prod_{a=1}^{L}F_{n^{(a)}}q^{Nn^{(a)}}\right)Z_{\vec{\zeta}'}(N)\,, \tag{3.51}$$

where we have shifted once again the eigenvalues as $\alpha^{(a)} \to \alpha^{(a)} + \sum_{b=1}^{a-1}u^{(b)}$. $Z_{\vec{\zeta}'}(N)$ is exactly the same as (3.31) where now the subscript combines the FI parameters and Fourier modes as

$$\zeta'^{(a)} = n^{(a)} - n^{(L)} + \sum_{b=1}^{a}\zeta^{(b)}\,. \tag{3.52}$$

The analysis proceeds as in the $SU(N)$ case, where now the Kronecker delta in (3.39) requires the sum of the FI parameters to vanish $\zeta'^{(L)} = \sum_a \zeta^{(a)} = 0$. Likewise the grand partition function (3.43) is the same with $\vec{n}$ replaced by $\vec{\zeta}'$.

The main difference from the $SU(N)$ case lies in the highly non-trivial Fourier coefficients in (3.50) and the sum over $\Xi_{\vec{\zeta}'}$ weighted by those coefficients is much harder to implement than the $SU(N)$ case, which we now return to.

[4]Note that the numerator of (3.44) can also be factored algebraically for $L = 4$, but we haven't investigated this case.

3.4 Large N results for long quivers

In this section we compute the Schur index for $SU(N)$ circular quivers with L nodes in the large N limit and with the product of flavour fugacities set to 1, so that $U = 0$. For quiver theories of arbitrary length, the result scales as N^0, is independent of the flavour fugacities, and there are no perturbative $1/N$ corrections. We address the exponential corrections in N for $L = 1$ and $L = 2$ in the next section. In the subsections below we present two different methods which both give the same perturbatively exact large N result.

3.4.1 Asymptotics from the grand canonical partition function

The first method relies on the expression (3.46) for the grand canonical partition function $\Xi_{\vec{n}}$ in terms of the roots of a degree $2L$ polynomial. We solve for the roots of the polynomial at large μ, from that obtain $\Xi_{\vec{n}}$ and through (3.41) find $Z_{\vec{n}}$. Taking the sum over the Fourier modes (3.30) and including the prefactors in (3.27), we finally obtain the index up to non-perturbative corrections in the large N limit.

We first compute the large μ expansion of the d_j, introduced in (3.45). Recall that $X_j = -e^{-2id_j}$ are the roots of the polynomial

$$\prod_{a=1}^{L}\left(1 + q^{-2n^{(a)}}X^2\right) + \kappa q^{-n}X^L\,. \tag{3.53}$$

X_j can be expanded at large κ as

$$X_j = X_j^{(0)}\kappa^{\gamma_j}(1 + \mathcal{O}(\kappa^{\beta_j}))\,, \tag{3.54}$$

where $X_j^{(0)}$ is a (non zero) constant and $\beta_j < 0$. Plugging this ansatz into (3.53), and expanding at leading order in κ, the roots must satisfy

$$\begin{cases} 0 = q^{-2n}(X_j^{(0)})^{2L}\kappa^{2\gamma_j L} + q^{-n}\kappa^{\gamma_j L+1}(X_j^{(0)})^L + \mathcal{O}(\kappa^{2\gamma_j(L-1)})\,, & \gamma_j > 0, \\ 0 = (X_j^{(0)})^L q^{-n}\kappa + \mathcal{O}(\kappa^0)\,, & \gamma_j = 0\,, \\ 0 = 1 + \kappa^{1-\gamma_j L}q^{-n}(X_j^{(0)})^L + \mathcal{O}(\kappa^{-2\gamma_j})\,, & \gamma_j < 0\,. \end{cases} \tag{3.55}$$

The second line has no solutions, while the first and third lines each admit L solutions with $\gamma_j = \frac{1}{L}$ and $\gamma_j = -\frac{1}{L}$ respectively, and with

$$X_j^{(0)} = e^{\frac{i\pi(2j+1)}{L}}q^{\frac{n}{L}}\,, \qquad j = 1, \cdots, L\,. \tag{3.56}$$

Going to the next order, we find that for all of the roots (3.54) $\beta_j = -\frac{2}{L}$. We can then readily deduce the large μ expansions for d_j

$$d_{j,\pm} = \pm\frac{i\mu}{2L} + \frac{(L+1-2j)\pi}{2L} + \frac{n\pi\tau}{2L} + \mathcal{O}\big(\kappa^{-\frac{2}{L}}\big)\,, \qquad j = 1, \cdots, L\,, \tag{3.57}$$

where the indices differ slightly from the ones used in (3.45). Using the above expression, we can in turn expand (3.46) in the large μ limit as

$$\begin{aligned}\Xi_{\vec{n}} = \frac{q^{\sum_{a=1}^{L}(n^{(a)})^2}}{\vartheta_4^L\vartheta_3^L}\prod_{j=1}^{L}\vartheta_3\Big(\frac{i\mu}{2L} + \frac{(L+1-2j)\pi}{2L} + \frac{n\pi\tau}{2L}, q^{\frac{1}{2}}\Big)\\ \times\vartheta_3\Big(\frac{i\mu}{2L} + \frac{(L+1-2j)\pi}{2L} - \frac{n\pi\tau}{2L}, q^{\frac{1}{2}}\Big) + \mathcal{O}(\kappa^{-2/L})\,.\end{aligned} \tag{3.58}$$

This last expression involves the product of theta functions shifted by fractions of π. This product can be done using then the identity (E.14) proven in appendix E, and we obtain

$$\Xi_{\vec{n}} = \frac{q^{\sum_{a=1}^{L}(n^{(a)})^2}}{\vartheta_4^L\vartheta_3^L}\frac{\eta^{2L}(\frac{\tau}{2})}{\eta^2(\frac{L\tau}{2})}\vartheta_3\Big(\frac{i\mu}{2} + \frac{n\pi\tau}{2}, q^{\frac{L}{2}}\Big)\vartheta_3\Big(\frac{i\mu}{2} - \frac{n\pi\tau}{2}, q^{\frac{L}{2}}\Big) + \mathcal{O}(\kappa^{-2/L})\,. \tag{3.59}$$

Now using Watson's identities (E.9), as well as (E.8), gives

$$\begin{aligned}\Xi_{\vec{n}} = q^{\sum_{a=1}^{L}(n^{(a)})^2}\frac{\eta^L(\tau)}{\vartheta_3^L\vartheta_4(0,q^L)\eta(L\tau)}\big(\vartheta_3(i\mu,q^L)\vartheta_3(n\pi\tau,q^L) + \vartheta_2(i\mu,q^L)\vartheta_2(n\pi\tau,q^L)\big)\\ + \mathcal{O}(\kappa^{-2/L})\,.\end{aligned} \tag{3.60}$$

From this expression one can obtain $Z_{\vec{n}}$ (3.31) via the integral transform (3.41) and the expressions for the Fourier coefficients of the theta function (E.5)

$$\begin{aligned}Z_{\vec{n}}(N) = q^{\sum_{a=1}^{L}(n^{(a)})^2}q^{\frac{LN^2}{4}}\frac{\eta^L(\tau)}{2\vartheta_3^L\vartheta_4(0,q^L)\eta(L\tau)}\\ \times\big((1+(-1)^N)\vartheta_3(\pi\tau n,q^L) + (1-(-1)^N)\vartheta_2(\pi\tau n,q^L)\big) + \dots\end{aligned} \tag{3.61}$$

To get the index we need to sum over the Fourier modes $\vec{n}$, as in (3.30) (recall that $n^{(L)} = 0$ (3.39)). Using the series representation (E.2) of the theta functions above, we find

$$\begin{aligned}
&\sum_{\vec{n}\in\mathbb{Z}^{L-1}} q^{\sum_{a=1}^{L-1}(n^{(a)})^2} e^{-2iN\sum_{a=1}^{L-1} n^{(a)}u^{(a)}} \vartheta_3(\pi\tau n, q^L) \\
&\quad = \sum_{\{\vec{n},p\}\in\mathbb{Z}^L} q^{\sum_{a=1}^{L-1}(n^{(a)})^2+Lp^2} e^{-2iN\sum_{a=1}^{L-1} n^{(a)}u^{(a)}} \\
&\quad = \sum_{\{\vec{n},p\}\in\mathbb{Z}^L} q^{\sum_{a=1}^{L-1}(n^{(a)}+p)^2+p^2} e^{-2iN\sum_{a=1}^{L-1}(n^{(a)}+p)u^{(a)}-2iNpu^{(L)}} = \prod_{a=1}^{L} \vartheta_3(Nu^{(a)})\,.
\end{aligned} \tag{3.62}$$

Similarly we obtain

$$\sum_{\vec{n}\in\mathbb{Z}^{L-1}} q^{\sum_{a=1}^{L-1}(n^{(a)})^2} e^{-2iN\sum_{a=1}^{L-1} n^{(a)}u^{(a)}} \vartheta_2(\pi\tau n, q^L) = \prod_{a=1}^{L} \vartheta_2(Nu^{(a)})\,. \tag{3.63}$$

The sum over $Z_{\vec{n}}$ is now simple, with only some care required to account for the $(-1)^N$ factors. For this, we use the formula (see (E.4))

$$q^{\frac{N^2}{4}} \vartheta_3\Big(Nu^{(a)} + \frac{\pi\tau}{2}N\Big) = \begin{cases} q^{-iN^2u^{(a)}}\vartheta_3(Nu^{(a)})\,, & N\,\text{even}, \\ q^{-iN^2u^{(a)}}\vartheta_2(Nu^{(a)})\,, & N\,\text{odd}, \end{cases} \tag{3.64}$$

which gives

$$Z(N) = q^{\frac{LN^2}{2}} \frac{\eta^L(\tau)}{\vartheta_4(0,q^L)\eta(L\tau)} \prod_{a=1}^{L} \frac{\vartheta_3\left(Nu^{(a)} + \frac{\pi\tau}{2}N\right)}{\vartheta_3} + \dots \tag{3.65}$$

Substituting this result in (3.27), we finally obtain

$$\mathcal{I}_{SU(N)} = \frac{q^{\frac{L}{6}}}{\vartheta_4(0,q^L)\eta^L(\tau)\eta(L\tau)} + \mathcal{O}\big(e^{-cN}\big)\,, \qquad c > 0\,. \tag{3.66}$$

Writing the remaining theta function in terms of eta functions this can also be written as

$$\mathcal{I}_{SU(N)} = \frac{q^{\frac{L}{6}}}{\eta^L(\tau)\eta^2(\frac{L\tau}{2})} + \mathcal{O}\big(e^{-cN}\big)\,. \tag{3.67}$$

As previously mentioned, for the case of $L = 1$ the result is slightly modified because the matter multiplet is in the adjoint rather than bi-fundamental representation (see footnote 3). (3.67) is the main result of this section, which we reproduce in the next subsection using different techniques. We find that the full perturbative dependence on N is given by this constant term with no subleading $1/N$ corrections (ignoring non-perturbative corrections), and that the result does not depend on the flavour fugacities. We expect the leading large N result for $U(N)$ quivers to be of a similar form, since the index is a series in powers of q with integer coefficients. It would be interesting to study this case by including the Fourier coefficients in (3.50).

To get to the final result we have first integrated over μ and then summed over $\vec{n}$. For completeness we do it also in the reverse order, first summing $\Xi_{\vec{n}}$ over the Fourier modes. This suggests to define an overall $\Xi(\kappa)$ as

$$\begin{aligned}\Xi(\kappa) &\equiv 1+\sum_{\vec{n}\in\mathbb{Z}^{L-1}} e^{-2iN\sum_{a=1}^{L-1}u^{(a)}n^{(a)}}\sum_{N=1}^{\infty} Z_{\vec{n}}(N)\kappa^N \\ &= 1+\sum_{\vec{n}\in\mathbb{Z}^{L-1}} e^{-2iN\sum_{a=1}^{L-1}u^{(a)}n^{(a)}}\left(\Xi_{\vec{n}}(\kappa)-1\right).\end{aligned} \tag{3.68}$$

Using (3.60) and (3.62) we obtain

$$\Xi(\kappa)=\frac{\eta^L(\tau)}{\eta^2(\frac{L\tau}{2})}\left(\vartheta_3(i\mu,q^L)\prod_{a=1}^{L}\frac{\vartheta_3(Nu^{(a)})}{\vartheta_3}+\vartheta_2(i\mu,q^L)\prod_{a=1}^{L}\frac{\vartheta_2(Nu^{(a)})}{\vartheta_3}\right)+\ldots \tag{3.69}$$

Furthermore we can define the odd and even parts of Ξ as

$$\Xi_{\pm}(\kappa)=\frac{1}{2}\left(\Xi(\kappa)\pm\Xi(-\kappa)\right). \tag{3.70}$$

Since ϑ_3 is periodic in π and ϑ_2 antiperiodic, we find

$$\begin{aligned}\Xi_{+}(\kappa) &= \frac{\eta^L(\tau)}{\eta^2(\frac{L\tau}{2})}\vartheta_3(i\mu,q^L)\prod_{a=1}^{L}\frac{\vartheta_3(Nu^{(a)})}{\vartheta_3}+\ldots \\ \Xi_{-}(\kappa) &= \frac{\eta^L(\tau)}{\eta^2(\frac{L\tau}{2})}\vartheta_2(i\mu,q^L)\prod_{a=1}^{L}\frac{\vartheta_2(Nu^{(a)})}{\vartheta_3}+\ldots\end{aligned} \tag{3.71}$$

Recall the factor in (3.27) relating the index with the rescaled index $Z(N)$

$$\frac{q^{-\frac{N^2L}{2}}q^{\frac{L}{6}}q^{iN^2\sum_a u^{(a)}}}{\eta^{2L}(\tau)}\prod_{a=1}^{L}\frac{\vartheta_3}{\vartheta_3\left(Nu^{(a)}+N\frac{\pi\tau}{2}\right)}, \tag{3.72}$$

which due to (3.64) has a nice alternating behavior between even and odd N apart from a factor of $q^{-\frac{N^2L}{4}}$. This suggest that we can also define a *grand index* as

$$\hat{\Xi}(\kappa)=1+\sum_{N=1}^{\infty} q^{\frac{LN^2}{4}}\mathcal{I}_{SU(N)}(N)\kappa^N. \tag{3.73}$$

This does not involve all the rescaling factors in (3.27), and the difference between even and odd N is captured by different rescalings of the $\Xi_\pm$ defined above as

$$\begin{aligned}\hat{\Xi}(\kappa) &= \Xi_+(\kappa)\,\frac{q^{\frac{L}{6}}}{\eta^{2L}(\tau)}\prod_{a=1}^{L}\frac{\vartheta_3}{\vartheta_3(Nu^{(a)})} + \Xi_-(\kappa)\,\frac{q^{\frac{L}{6}}}{\eta^{2L}(\tau)}\prod_{a=1}^{L}\frac{\vartheta_3}{\vartheta_2(Nu^{(a)})} \\ &= \frac{q^{\frac{L}{6}}}{\eta^{L}(\tau)\eta^2(\frac{L\tau}{2})}\big(\vartheta_3(i\mu,q^L)+\vartheta_2(i\mu,q^L)\big)+\dots\end{aligned} \tag{3.74}$$

Equation (3.67) is easily reproduced from the inverse of (3.73), *i.e.*, the Fourier expansion of $\hat{\Xi}$.

3.4.2 Asymptotics from the grand potential

In the previous subsection we used the formula for $\Xi_{\vec{n}}$ in terms of the roots of a polynomial (3.46) and used the large μ expansion of the roots to find (3.60), from which we deduced the perturbative part of the large N behavior of the index.

We now present an alternative way of obtaining the large μ limit of $\Xi_{\vec{n}}$ (3.60), by applying the large μ approximation to the grand potential (3.42). An analogue method was used in the case of 3-dimensional theories (see (2.183) and the surrounding discussion) and is instructive as it does not rely on the exact expression for $Z_{\vec{n};\ell}$, which may not be available in other settings.

To find the grand potential at large μ we only need the asymptotic behavior of $Z_{\vec{n};\ell}$ at large ℓ. Following [85], we use the Mellin-Barnes representation

$$J_{\vec{n}}(\mu) = -\int_{c-i\infty}^{c+i\infty}\frac{d\ell}{2\pi i}\,\frac{\pi}{\sin\pi\ell}\,\frac{Z_{\vec{n};\ell}}{\ell}e^{\ell\mu}\,,\quad 0<c<1\,, \tag{3.75}$$

and extract the leading order in the large μ from the poles of $Z_{\vec{n};\ell}$ with largest $\mathrm{Re}(\ell)$.

The representation (3.75) requires some explanation and justification. We first write $Z_{\vec{n};\ell}$ (3.39) as an analytic function of ℓ by splitting it into two sums, one for positive p and one for strictly negative p. Denoting the sum over the terms with positive p as $Z^+_{\vec{n};\ell}$, we have

$$\begin{aligned}Z^+_{\vec{n};\ell} &= \sum_{p=0}^{\infty}\frac{q^{\ell\sum_{a=1}^{L}(p-n^{(a)}+\frac12)}}{\prod_{a=1}^{L}\big(1+q^{2(p-n^{(a)}+\frac12)}\big)^{\ell}} \\ &= q^{-\ell\sum_{a=1}^{L}(n^{(a)}-\frac12)}\sum_{p=0}^{\infty}q^{\ell Lp}\sum_{\vec{k}\in\mathbb{Z}_+^L}q^{2\sum_{a=1}^{L}k^{(a)}(p-n^{(a)}+\frac12)}\prod_{a=1}^{L}\binom{-\ell}{k^{(a)}} \\ &= q^{-\ell\sum_{a=1}^{L}(n^{(a)}-\frac12)}\sum_{\vec{k}\in\mathbb{Z}_+^L}q^{-2\sum_{a=1}^{L}k^{(a)}(n^{(a)}-\frac12)}\prod_{a=1}^{L}\binom{-\ell}{k^{(a)}}\sum_{p=0}^{\infty}q^{p(\ell L+2\sum_{a=1}^{L}k^{(a)})}\,.\end{aligned} \tag{3.76}$$

Doing the summation over p, we obtain

$$Z^{+}_{\vec{n};\ell} = \sum_{\vec{k}\in\mathbb{Z}^L_+} \frac{q^{-\sum_{a=1}^{L}(2k^{(a)}+\ell)(n^{(a)}-\frac{1}{2})}}{1-q^{(\ell L+2\sum_{a=1}^{L}k^{(a)})}} \prod_{a=1}^{L}\binom{-\ell}{k^{(a)}} . \tag{3.77}$$

This final form admits an analytical continuation in ℓ to the complex plane, and a similar argument can be used for $Z^{-}_{\vec{n};\ell}$, which is obtained by replacing $n^{(a)} \to -n^{(a)}$. For negative values of μ one can then compute the right hand side of (3.75) by closing the contour with an infinite half circle enclosing the simple poles due to $\pi/\sin\pi\ell$ at positive values of ℓ, but none of the poles due to $Z_{\vec{n};\ell}$. Using the fact that

$$\operatorname*{Res}_{\ell=n} \frac{\pi}{\sin\pi\ell} = (-1)^n , \tag{3.78}$$

and the fact that the evaluation of the integral on the remaining part of the contour gives zero, we recover the representation (3.42) as an infinite sum, which is indeed convergent for negative μ.

To analytically continue $J_{\vec{n}}(\mu)$ to positive values of μ, we close the contour in (3.75) with an infinite half-circle in the $\mathrm{Re}(\ell) \leqslant c$ half-plane. In this enclosed region, the poles of $Z_{\vec{n};\ell}$ and of the cosecant are then at

$$\begin{aligned} \ell &= -\frac{2}{L}\sum_{a=1}^{L}k^{(a)} + \frac{2l}{L\tau}\,, \qquad && k^{(a)}\in\mathbb{N}\,, \quad l\in\mathbb{Z}\,, \\ \ell &= -n\,, && n\in\mathbb{N}\,. \end{aligned} \tag{3.79}$$

It can be shown that the contour integrals coming from the integration over the infinite half-circle do not contribute, so that (3.75) is determined only by the residues of the poles (3.79).

As explained in the previous section we are ultimately interested in $J_{\vec{n}}(\mu)$ for large μ. The poles that are not on the imaginary axis are exponentially suppressed in this limit. We can thus write

$$J_{\vec{n}}(\mu) = -\sum_{m\in\mathbb{Z}} \operatorname*{Res}_{\ell=\frac{2m}{L\tau}} \frac{\pi}{\sin\pi\ell}\frac{Z_{\vec{n};\ell}}{\ell}e^{\ell\mu} + \mathcal{O}(e^{-\frac{2\mu}{L}})\,, \tag{3.80}$$

where the scaling in μ of the next to leading order can be deduced from the lattices

(3.79). For the residue of the pole at $\ell = 0$, we obtain

$$\begin{aligned}
&-\operatorname*{Res}_{\ell=0} \frac{\pi}{\sin \pi\ell} \frac{Z_{\vec{n};\ell}}{\ell} e^{\ell\mu} \\
&= i\frac{4\pi^2 + L^2\pi^2\tau^2 + 12\mu^2 - 12\pi^2\tau^2 n^2}{12L\pi\tau} + \sum_{a=1}^{L}\sum_{k=1}^{\infty} \cosh 2i\pi\tau k n^{(a)} \frac{(-1)^k}{k \sinh -i\pi\tau k} \\
&= i\frac{4\pi^2 + L^2\pi^2\tau^2 + 12\mu^2 - 12\pi^2\tau^2 n^2}{12L\pi\tau} - \sum_{a=1}^{L}\left(\log \frac{\vartheta_3(\tau\pi n^{(a)})}{\vartheta_3} + \frac{i\pi\tau}{12} + \frac{1}{6}\log \frac{4}{kk'}\right),
\end{aligned} \tag{3.81}$$

where the sum over k was done using (E.10).

The sum over the poles on the imaginary axis but away from the origin gives

$$\begin{aligned}
-\sum_{m\neq 0} \operatorname*{Res}_{\ell=\frac{2m}{L\tau}} \frac{\pi}{\sin \pi\ell} \frac{Z_{\vec{n};\ell}}{\ell} e^{\ell\mu} &= \sum_{m\neq 0} (-1)^{m+1} \frac{e^{\frac{2m\mu}{L\tau}} \cos \frac{2\pi m n}{L}}{m \sinh \frac{2i\pi m}{L\tau}} \\
&= \log \frac{\vartheta_3\big(\frac{\pi}{L}\big(\frac{i\mu}{\pi\tau} + n\big), q'^{\frac{2}{L}}\big)\vartheta_3\big(\frac{\pi}{L}\big(\frac{i\mu}{\pi\tau} - n\big), q'^{\frac{2}{L}}\big)}{\vartheta_3^2\big(0, q'^{\frac{2}{L}}\big)} - \frac{i\pi}{3L\tau} + \frac{1}{3}\log \frac{4}{\tilde{k}\tilde{k}'}.
\end{aligned} \tag{3.82}$$

This sum was again done using (E.10) but with the complement nome and corresponding modulus

$$q' = e^{-\frac{i\pi}{\tau}}, \quad \tilde{k} = \frac{\vartheta_2^2(0, q'^{\frac{2}{L}})}{\vartheta_3^2(0, q'^{\frac{2}{L}})}, \quad \tilde{k}' = \frac{\vartheta_4^2(0, q'^{\frac{2}{L}})}{\vartheta_3^2(0, q'^{\frac{2}{L}})}. \tag{3.83}$$

Applying a modular transformation (E.6) to (3.82) gives

$$\log \frac{\vartheta_3\big(\frac{i\mu}{2} + \frac{\pi\tau n}{2}, q^{\frac{L}{2}}\big)\vartheta_3\big(\frac{i\mu}{2} - \frac{\pi\tau n}{2}, q^{\frac{L}{2}}\big)}{\vartheta_3^2\big(0, q^{\frac{L}{2}}\big)} - \frac{i\mu^2}{\pi\tau L} + \frac{i\pi\tau n^2}{L} - \frac{i\pi}{3L\tau} + \frac{1}{3}\log \frac{4}{\tilde{k}\tilde{k}'}. \tag{3.84}$$

Putting the contributions from all the poles on the imaginary axis together, we obtain

$$\begin{aligned}
J_{\vec{n}}(\mu) = \log &\frac{\vartheta_3\big(\frac{i\mu}{2} + \frac{\pi\tau n}{2}, q^{\frac{L}{2}}\big)\vartheta_3\big(\frac{i\mu}{2} - \frac{\pi\tau n}{2}, q^{\frac{L}{2}}\big)}{\vartheta_3^2\big(0, q^{\frac{L}{2}}\big)} \\
&+ \frac{2-L}{3}\log 2 + \frac{1}{3}\log \frac{(kk')^{L/2}}{\tilde{k}\tilde{k}'} - \sum_{a=1}^{L}\log \frac{\vartheta_3(\tau\pi n^{(a)})}{\vartheta_3} + \mathcal{O}(e^{-2\mu/L}).
\end{aligned} \tag{3.85}$$

Finally we use Watson's identity (E.9) to rewrite the product of theta functions in the first line in terms of theta functions with nome q^L. We also replace $k, k', \tilde{k}, \tilde{k}'$, applying modular transformations to the latter two. Then we use the quasi-periodicity of the theta function (E.4), and use (E.7) and (E.8) to express the result in terms of the Dedekind eta functions to find

$$
\begin{aligned}
J_{\vec{n}}(\mu) = \log \big(\vartheta_3(i\mu, q^L)\vartheta_3(\pi\tau n, q^L) + \vartheta_2(i\mu, q^L)\vartheta_2(\pi\tau n, q^L)\big) \\
+ \log \frac{\eta(\tau)^L}{\eta(L\tau)\vartheta_3^L\vartheta_4(0, q^L)} + i\pi\tau \sum_{a=1}^{L}(n^{(a)})^2 + \mathcal{O}(e^{-2\mu/L}) \,.
\end{aligned}
\tag{3.86}
$$

We finally obtain

$$
\begin{aligned}
\Xi_{\vec{n}}(\kappa) = q^{\sum_{a=1}^{L}(n^{(a)})^2} \frac{\eta(\tau)^L}{\vartheta_3^L\vartheta_4(0, q^L)\eta(L\tau)} \\
\times \big(\vartheta_3(i\mu, q^L)\vartheta_3(\pi\tau n, q^L) + \vartheta_2(i\mu, q^L)\vartheta_2(\pi\tau n, q^L)\big) + \mathcal{O}(\kappa^{-2/L}) \,,
\end{aligned}
\tag{3.87}
$$

which is identical to (3.60).

3.5 Exact large N expansions for short quivers

For quivers with one or two nodes we can compute the Schur index exactly, without having to resort to perturbative techniques. Recall that the grand partition function can be expressed by a product of theta functions (3.46) evaluated at the roots of the polynomial (3.44)

$$
X^L q^{-n}\, \kappa + \prod_{a=1}^{L} \big(1 + X^2 q^{-2n^{(a)}}\big) \,. \tag{3.88}
$$

For $L = 1$ and $L = 2$ this polynomial is quadratic in X and X^2 respectively, and so the roots are simply algebraic.[5] This results in completely explicit expressions for the grand partition functions which allows us to find closed form expressions for the indices of these theories. We carry out the analysis for the $L = 1$ and $L = 2$ $SU(N)$ theories below.

3.5.1 Single node, $\mathcal{N} = 4$ SYM

For the single node theory (with the only flavour fugacity $e^{2iu^{(1)}} = e^{2iU}$ set to one), the polynomial (3.88) can be factored as

$$
X\kappa + 1 + X^2 = \Big(1 + \tfrac{\kappa+\sqrt{\kappa^2-4}}{2}\, X\Big)\Big(1 + \tfrac{\kappa-\sqrt{\kappa^2-4}}{2}\, X\Big) \,. \tag{3.89}
$$

Comparing with (3.45) we readily obtain $d_\pm = \pm\frac{i}{2}\log\frac{\kappa+\sqrt{\kappa^2-4}}{2} = \pm\frac{1}{2}\arccos\frac{\kappa}{2}$.

[5] For the theory with $L = 4$ the polynomial is quartic and so can also be factored algebraically. It would be interesting to see if a similar analysis would also give a complete solution for the index of this theory.

Unlike the cases of $L > 1$, there are no Fourier modes to sum over, giving a single free Fermi gas whose grand partition function is[6] (3.46)

$$\Xi^{\mathcal{N}=4}(\kappa) = \frac{\vartheta_3^2\left(\frac{1}{2}\arccos\frac{\kappa}{2}, q^{\frac{1}{2}}\right)}{\vartheta_3\vartheta_4} = \frac{1}{\vartheta_4}\left[\vartheta_3\left(\arccos\frac{\kappa}{2}\right) + \frac{\vartheta_2}{\vartheta_3}\vartheta_2\left(\arccos\frac{\kappa}{2}\right)\right]. \tag{3.90}$$

Recall that in terms of the grand partition function, the index is given by[7]

$$\mathcal{I}^{\mathcal{N}=4}_{SU(N)}(N) = \frac{q^{-\frac{N^2}{2}}q^{\frac{1}{4}}\eta^2(\frac{\tau}{2})\vartheta_3}{\eta^4(\tau)\vartheta_3(\frac{\pi\tau}{2}N)}\int_{-i\pi}^{i\pi}\frac{d\mu}{2\pi i}\,e^{-\mu N}\Xi^{\mathcal{N}=4}(e^{\mu})\,. \tag{3.91}$$

Expanding the square of the theta function in the middle expression of (3.90) gives

$$e^{-\mu N}\Xi^{\mathcal{N}=4}(e^{\mu}) = \frac{e^{-\mu N}}{\vartheta_3\vartheta_4}\sum_{m=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}q^{\frac{1}{2}(m^2+j^2)}\left(\tfrac{e^{\mu}+\sqrt{e^{2\mu}-4}}{2}\right)^{m+j}. \tag{3.92}$$

Applying the expansion formula (G.2) this is

$$\frac{1}{\vartheta_3\vartheta_4}\sum_{m=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\sum_{s=0}^{\infty}q^{\frac{1}{2}(m^2+j^2)}e^{\mu(m+j-2s-N)}\frac{(-1)^s(m+j)(m+j-s-1)!}{s!(m+j-2s)!}. \tag{3.93}$$

Integrating over μ simply gives a Kronecker delta $\delta_{m+j-2s-N}$, which removes the sum over m

$$\begin{aligned} Z(N) &= \int_{-i\pi}^{i\pi}\frac{d\mu}{2\pi i}e^{-\mu N}\Xi^{\mathcal{N}=4}(e^{\mu}) \\ &= \frac{1}{\vartheta_3\vartheta_4}\sum_{j=-\infty}^{\infty}\sum_{s=0}^{\infty}q^{(j-s-\frac{N}{2})^2+(s+\frac{N}{2})^2}\frac{(-1)^s(N+2s)(N+s-1)!}{s!N!}. \end{aligned} \tag{3.94}$$

Finally evaluating the sum over j and including the prefactors from (3.91) yields

$$\begin{aligned} \mathcal{I}^{\mathcal{N}=4}_{SU(N)}(N) &= \frac{q^{\frac{1}{4}}}{\eta^3(\tau)}\sum_{s=0}^{\infty}(-1)^s(N+2s)\frac{(N+s-1)!}{s!N!}q^{Ns+s^2} \\ &= \frac{q^{\frac{1}{4}}}{\eta^3(\tau)}\sum_{s=0}^{\infty}(-1)^s\left[\binom{N+s}{N}+\binom{N+s-1}{N}\right]q^{Ns+s^2}. \end{aligned} \tag{3.95}$$

At leading order at large N this is simply $q^{1/4}/\eta^3(\tau)$, which differs from (3.67) by the contribution of one free hypermultiplet (see footnotes 3 and 7).

For the special case of $L = 1$, an essentially identical analysis works in the case

[6] Although the matrix model still has a delta function coming from the tracelessness condition of $SU(N)$, the Kronecker delta in (3.39) ensures that only the mode with $n = 0$ contributes.

[7] Note the additional factor of $q^{\frac{1}{12}}\eta^2(\frac{\tau}{2})\eta^{-2}(\tau)$ in (3.91) compared with (3.27) with vanishing u's, coming from the fact that the hypermultiplets are in adjoint rather than bi-fundamental representations of $SU(N)$ (see also footnote 3).

where the gauge group is $U(N)$ rather than $SU(N)$. In each case the index matrix model can be written in terms of the same unique free Fermi gas, and the only difference is in the overall prefactor. Thus we immediately can write down

$$\mathcal{I}^{\mathcal{N}=4}_{U(N)}(N) = \frac{\eta(\tau)}{\eta^2(\frac{\tau}{2})}\sum_{s=0}^{\infty}(-1)^s\left[\binom{N+s}{N}+\binom{N+s-1}{N}\right]q^{Ns+s^2}. \tag{3.96}$$

3.5.2 Two nodes

For the two-node quiver, $L=2$, the polynomial (3.88) can be factored as

$$\kappa q^{-n}X^2+(1+X^2)(1+q^{-2n}X^2) = \left(1+\tfrac{\tilde\kappa+\sqrt{\tilde\kappa^2-4}}{2}\,q^{-n}X^2\right)\left(1+\tfrac{\tilde\kappa-\sqrt{\tilde\kappa^2-4}}{2}\,q^{-n}X^2\right), \tag{3.97}$$

where $\tilde\kappa=\kappa+q^n+q^{-n}$. Comparing with (3.47) we obtain $\tilde d_\pm=-\frac{\pi\tau n}{2}\pm\frac{1}{2}\arccos\frac{\tilde\kappa}{2}$ and substituting into (3.48) gives

$$\Xi^{L=2}_n(\kappa) = \frac{q^{n^2}}{\vartheta_3^2}\vartheta_3\Big(\frac{\pi\tau n}{2}+\frac{1}{2}\arccos\frac{\tilde\kappa}{2}\Big)\vartheta_3\Big(\frac{\pi\tau n}{2}-\frac{1}{2}\arccos\frac{\tilde\kappa}{2}\Big). \tag{3.98}$$

Using Watson's identity (E.9), this can also be written as the sum of theta functions with nome q^2, (*c.f.*, the last expression in (3.90)), but this representation will not be simpler for us.

In terms of the grand partition function, the index with flavour fugacities such that $u\equiv u^{(1)}=-u^{(2)}$, is given by (see (3.27), (3.30) and (3.41))

$$\mathcal{I}^{L=2}(N) = \frac{q^{-N^2}q^{\frac{1}{3}}\vartheta_3^2}{\eta^4(\tau)\prod_\pm\vartheta_3(N\frac{\pi\tau}{2}\pm Nu)}\sum_{n=-\infty}^{\infty}\frac{e^{-2iNnu}}{2\pi i}\int_{-i\pi}^{i\pi}d\mu\, e^{-\mu N}\Xi^{L=2}_n(e^\mu). \tag{3.99}$$

One could proceed by evaluating the large μ expansion of the integrand, but it turns out to be simpler to perform instead the full expansion of the grand partition function in powers of e^μ and q.

Expanding the theta functions in (3.98), the integrand of (3.99) can be written as

$$e^{-\mu N}\Xi^{L=2}_n = \frac{e^{-\mu N}}{\vartheta_3^2}\sum_{m=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}q^{n^2+m^2+j^2}q^{n(m+j)}\left(\tfrac{e^\mu+q^n+q^{-n}+\sqrt{(e^\mu+q^n+q^{-n})^2-4}}{2}\right)^{m-j}. \tag{3.100}$$

Using the expansion formula (G.1) this is

$$\begin{aligned}\frac{1}{\vartheta_3^2}&\sum_{m=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}q^{n^2+m^2+j^2}q^{n(m+j+k-l)}e^{\mu(m-j-k-l-N)}\\ &\times(m-j)\frac{(m-j-k-1)!(m-j-l-1)!}{k!l!(m-j-k-l)!(m-j-k-l-1)!}.\end{aligned} \tag{3.101}$$

Integrating over μ gives a Kronecker delta $\delta_{m-j-k-l-N}$, which removes the sum over m

$$Z_n = \int_{-i\pi}^{i\pi} \frac{d\mu}{2\pi i}\, e^{-\mu N}\Xi_n = \frac{1}{\vartheta_3^2} \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} q^{n^2+(j+k+l+N)^2+j^2} q^{n(2j+2k+N)} \times (N+l+k)\frac{(N+k-1)!(N+l-1)!}{N!(N-1)!k!l!}\,. \tag{3.102}$$

Summing over n (3.30) then gives

$$Z(N) = \sum_{n=-\infty}^{\infty} e^{-2iNnu} Z_n = \frac{\vartheta_3(\frac{\pi\tau}{2}N - uN)}{\vartheta_3^2} \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} e^{2iuN(j+k)} \times q^{(j+k+l+N)^2+j^2-(j+k)(j+k+N)}(N+l+k)\frac{(N+k-1)!(N+l-1)!}{N!(N-1)!k!l!}\,. \tag{3.103}$$

Finally evaluating the sum over j and including the prefactors from (3.99) we obtain

$$\mathcal{I}^{L=2}(N) = \frac{q^{\frac{1}{3}}}{\eta^4(\tau)} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(N+k+l)\frac{(N+k-1)!(N+l-1)!}{N!(N-1)!k!l!} q^{N(k+l)+2kl} e^{2iuN(k-l)}\,. \tag{3.104}$$

Alternatively this can be written as

$$\frac{q^{\frac{1}{3}}}{\eta^4(\tau)} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[\binom{N+k}{N}\binom{N+l-1}{N-1} + \binom{N+k-1}{N-1}\binom{N+l-1}{N}\right] \times q^{N(k+l)+2kl} e^{2iuN(k-l)}\,. \tag{3.105}$$

At leading order at large N this is simply

$$\mathcal{I}^{L=2}(N) = \frac{q^{\frac{1}{3}}}{\eta^4(\tau)} + \mathcal{O}(q^N)\,, \tag{3.106}$$

in agreement with (3.67). Here we see explicitly how the dependence on u appears from terms in the sum with $k-l \neq 0$, all of which are exponentially suppressed at large N.

As in the case of $\mathcal{N}=4$ SYM in the previous section, the large N expansion (3.104) begs for a holographic interpretation (at least for $u=0$). For $\mathcal{N}=4$ there is a single sum (3.95) while here there is a double sum. In both cases the leading exponential term is proportional to N, suggesting a D3-brane interpretation. The double sum could correspond to two different types of D3 giant gravitons, with the extra $2kl$ term signifying some interaction between the two stacks of branes. It would be interesting to find appropriate supergravity solutions and/or brane embeddings that would reproduce this structure.

3.6 Finite N results for short quivers

In the previous section we showed how to obtain an exact, all order large N expansions for the indices of circular quivers of one or two nodes. Here we consider the indices of short quivers for finite values of N, and show how to obtain exact expressions for them in terms of elliptic functions. One obvious approach would be to take the large N expansions (3.95) and (3.104) and try to resum them for finite values of N, which is an approach we employ in section 3.6.3. However, it can also be instructive to take as a starting point expressions that appeared earlier in the Fermi gas formulation, such as the spectral traces (3.39) or the grand partition function (3.46). Such approaches may prove to be more easily generalised to cases where all order large N results such as (3.95) and (3.104) are lacking. Thus we present a number of methods, commenting on their applicability to the examples we consider.

3.6.1 Evaluating the spectral traces

Here we discuss the possibility of computing exactly the spectral traces (3.39), which in the case without flavour fugacities becomes.

$$Z_{\vec{n};\ell} = \delta_{n^{(L)}} \sum_{p\in\mathbb{Z}} \prod_{a=1}^{L} \left(\frac{1}{2\cosh i\pi\tau\big(p - n^{(a)} - \frac{1}{2}\big)} \right)^{\ell} . \tag{3.107}$$

Indeed for $L = 1$ (or $\mathcal{N} = 4$ SYM) they are particularly simple

$$Z_{\ell}^{\mathcal{N}=4} = \sum_{p\in\mathbb{Z}} \left(\frac{1}{2\cosh i\pi\tau(p - \frac{1}{2})} \right)^{\ell} . \tag{3.108}$$

Hyperbolic sums such as these were the subject of a paper by Zucker [92], who showed how to evaluate them in terms of elliptic integrals. We review the algorithm in some detail, since ingredients of it will be important for us in later sections

For even $\ell = 2s$ one has

$$
\begin{aligned}
Z_{2s}^{\mathcal{N}=4} &= \sum_{p\in\mathbb{Z}} \left(\frac{1}{2\cosh i\pi\tau(p-\frac{1}{2})}\right)^{2s} \\
&= 2\sum_{p=1}^{\infty} \left(q^{p-\frac{1}{2}} + q^{-p+\frac{1}{2}}\right)^{-2s} \\
&= 2\sum_{p=1}^{\infty}\sum_{n=0}^{\infty} \binom{-2s}{n} q^{(p-\frac{1}{2})(2s+2n)} \\
&= \frac{2}{(2s-1)!}\sum_{n=0}^{\infty}(-1)^n \frac{(2s+n-1)!}{n!}\frac{q^{s+n}}{1-q^{2s+2n}}
\end{aligned}
\tag{3.109}
$$

The ratio of factorials can be expanded as

$$
\frac{(2s+n-1)!}{n!} = \sum_{m=0}^{s-1} \alpha_m(s)(s+n)^{2s-2m-1}, \tag{3.110}
$$

where $\alpha_m(s)$ are numerical coefficients generated by

$$
\sum_{m=0}^{s-1} \alpha_m(s)t^m = \prod_{j=1}^{s-1}(1-j^2t)\,, \tag{3.111}
$$

(3.109) can then be written as

$$
Z_{2s} = 2\frac{(-1)^{s+1}}{(2s-1)!}\sum_{m=1}^{s-1} \alpha_m(s) D_{2s-2m-1}\,, \tag{3.112}
$$

where

$$
D_{2s+1} = \sum_{n=1}^{\infty} n^{2s+1}\frac{q^n}{1-q^{2n}}\,. \tag{3.113}
$$

The infinite sums D_{2s+1} appeared in the expansions of Jacobi elliptic functions found by Jacobi [93], and they can thus be generated by

$$
\sum_{s=0}^{\infty} D_{2s+1}\frac{(-4t^2)^s}{(2s)!} = \frac{EK}{2\pi^2} - \frac{K^2}{2\pi^2}(1-k^2)\,\mathrm{nd}^2\left(\frac{2Kt}{\pi}\right), \tag{3.114}
$$

where $K \equiv K(k^2)$ and $E \equiv E(k^2)$ are complete elliptic integrals of the first and second kind respectively, nd is an elliptic function (likewise cd below) and the elliptic modulus is $k = \vartheta_2^2/\vartheta_3^2$.

Similarly for odd $\ell = 2s+1$ the expansion

$$
\frac{(2r+k)!}{k!} = 2^{-2r}\sum_{m=0}^{r} \tilde{\alpha}_m(r)(2r+2k+1)^{2r-2m}\,, \tag{3.115}
$$

where $\tilde{\alpha}$ are generated by

$$\sum_{m=0}^{r} \tilde{\alpha}_m(r) t^m = \prod_{j=1}^{r} \left(1 - (2j-1)^2 t\right), \tag{3.116}$$

allows us to write

$$Z_{2s+1} = \frac{(-1)^s}{2^{2s-1}(2s)!} \sum_{m=0}^{s} \tilde{\alpha}_m(s) J_{2s-2m}, \tag{3.117}$$

where the infinite sums

$$J_{2s} = \sum_{n=1}^{\infty} (-1)^{n+1}(2n-1)^{2s} \frac{q^{n-\frac{1}{2}}}{1-q^{2n-1}} \tag{3.118}$$

are generated by

$$\sum_{s=0}^{\infty} J_{2s} \frac{(-t^2)^s}{(2s)!} = \frac{Kk}{2\pi} \operatorname{cd}\left(\frac{2Kt}{\pi}\right). \tag{3.119}$$

Using this algorithm, we find for $\ell = 1, 2, 3, 4$

$$\begin{aligned} Z_1 &= \frac{kK}{\pi}, & Z_2 &= \frac{KE - (1-k^2)K^2}{\pi^2}, \\ Z_3 &= \frac{Z_1}{8} - \frac{(1-k^2)kK^3}{2\pi^3}, & Z_4 &= \frac{Z_2}{6} - \frac{(1-k^2)k^2K^4}{3\pi^4}. \end{aligned} \tag{3.120}$$

Plugging these expressions into (3.34) and (3.27) (with the additional factor from footnote 3) gives

$$\begin{aligned} \mathcal{I}^{\mathcal{N}=4}(1) &= \frac{\eta^2(\frac{\tau}{2})\sqrt{k}}{\eta^4(\tau)} \frac{K}{\pi}, \qquad \mathcal{I}^{\mathcal{N}=4}(2) = \frac{q^{-\frac{3}{4}}\eta^2(\frac{\tau}{2})}{\eta^4(\tau)} \frac{-EK + K^2}{2\pi^2}, \\ \mathcal{I}^{\mathcal{N}=4}(3) &= \frac{q^{-2}\eta^2(\frac{\tau}{2})\sqrt{k}}{\eta^4(\tau)} \left(\frac{-3EK^2 + (2-k^2)K^3}{6\pi^3} + \frac{K}{24\pi} \right), \\ \mathcal{I}^{\mathcal{N}=4}(4) &= \frac{q^{-\frac{15}{4}}\eta^2(\frac{\tau}{2})}{\eta^4(\tau)} \left(\frac{3E^2K^2 - 6EK^3 + (3-2k^2)K^2}{24\pi^4} - \frac{EK - K^2}{24\pi^2} \right), \end{aligned} \tag{3.121}$$

This algorithm can be easily pushed to larger values of N and is easy to implement in Mathematica.

For quivers with more than a single node, we find that computing the spectral traces becomes intractable, due to the nontrivial dependence of (3.107) on the Fourier modes.

3.6.2 Expanding the grand partition function

Recall that the grand partition function is defined as (3.40)

$$\Xi_{\vec{n}}(\kappa) = 1 + \sum_{N=1}^{\infty} Z_{\vec{n}}(N)\kappa^N \,. \tag{3.122}$$

Since for $L = 1$ and $L = 2$ we have found closed expressions (3.90) and (3.98) for the left hand side, we can recover $Z_{\vec{n}}(N)$ for finite values of N by taking the Taylor expansion in κ and reading off the relevant coefficient. The index is then given by the sum (3.30) together with the prefactors from (3.27).

$\mathcal{N} = 4$ SYM

For $L = 1$, there are no fourier modes to sum over, and the coefficients of the κ expansion give immediately the rescaled index $Z(N)$. For example, expanding the rightmost expression in (3.90), the κ^2 coefficient gives[8]

$$Z(2) = \frac{\vartheta_4''}{8\vartheta_4} = \frac{-EK + K^2}{2\pi^2} \tag{3.123}$$

Including the prefactors in (3.27) combined with those from footnote 7 we immediately recover the same result as in (3.121). This approach can be straightforwardly pushed to higher values of N.

$L = 2$

for the two node case, there is still a sum over fourier modes to compute after extracting the $Z_n(N)$ coefficients. Indeed (3.27) specialised to $L = 2$ and without flavour fugacities becomes

$$\mathcal{I}^{L=2}(N) = \frac{q^{-N^2} q^{\frac{1}{3}} \vartheta_3^2}{\eta^4(\tau)\vartheta_3^2(\frac{N\pi\tau}{2})} \sum_{n=-\infty}^{\infty} Z_n^{L=2}(N) \,. \tag{3.124}$$

Expanding (3.98), the coefficient of κ^2 gives

$$Z_n^{L=2}(2) = \frac{q^{n^2}}{32}\left(\frac{\vartheta_3''\vartheta_3(n\pi\tau)}{\vartheta_3^2 \sin^2 n\pi\tau} + \frac{\sin n\pi\tau\vartheta_3''(n\pi\tau) - 2\cos n\pi\tau\vartheta_3'(n\pi\tau)}{\vartheta_3 \sin^3 n\pi\tau}\right) . \tag{3.125}$$

[8]To get the second equality, one can apply the heat equation satisfied by all Jacobi theta functions and convert the derivatives into τ derivatives. Then one can apply the standard relation $\vartheta_4 = \sqrt{\frac{2k'K}{\pi}}$ together with (3.139) to reduce everything to complete elliptic integrals.

For $n = 0$ this is

$$\begin{aligned} Z_0^{L=2}(2) &= \frac{1}{192\vartheta_3}\Big(4\vartheta_3\vartheta_3'' + 3\vartheta_3''^2 + \vartheta_3\vartheta_3^{(4)}\Big) \\ &= \frac{3E^2K^2 - 6(1-k^2)EK^3 + (1-k^2)(3-2k^2)K^4}{6\pi^4} + \frac{(1-k^2)K^2 - KE}{12\pi^2} \,. \end{aligned} \tag{3.126}$$

To get the expression in the last line, we again used the method of footnote 8.

For $n \neq 0$ the partition function (3.125) is

$$Z_{n\neq 0}^{L=2}(2) = \frac{\vartheta_3''}{\vartheta_3}\frac{1}{16\sin^2 n\pi\tau} + \frac{in\cos n\pi\tau - n^2\sin n\pi\tau}{8\sin^3 n\pi\tau}\,, \tag{3.127}$$

where we have used

$$\vartheta_3(n\pi\tau) = q^{-n^2}\vartheta_3\,, \qquad \vartheta_3'(n\pi\tau) = -2inq^{-n^2}\vartheta_3\,, \qquad \vartheta_3''(n\pi\tau) = q^{-n^2}(\vartheta_3'' - 4n^2\vartheta_3)\,. \tag{3.128}$$

The sum over n of the first term in (3.127) has been evaluated in [92]

$$\sum_{n\neq 0}\frac{1}{16\sin^2 n\pi\tau} = \frac{K}{12\pi^2}\big(3E - (2-k^2)K\big) - \frac{1}{48}\,. \tag{3.129}$$

The prefactor can be written in terms of elliptic integrals as

$$\frac{\vartheta_3''}{\vartheta_3} = \frac{-4KE + 4(1-k^2)K^2}{\pi^2}\,. \tag{3.130}$$

The sum over $n \neq 0$ of the second term vanishes since[9]

$$\sum_{n\neq 0}\frac{in\cos n\pi\tau}{8\sin^3 n\pi\tau} = \sum_{n\neq 0}\frac{n^2}{8\sin^2 n\pi\tau}\,. \tag{3.131}$$

Putting this together we obtain

$$\mathcal{I}^{L=2}(2) = \frac{q^{-\frac{5}{3}}}{\eta^4(\tau)}\frac{-3E^2K^2 + 2(2-k^2)EK^3 - (1-k^2)K^4}{6\pi^4}\,, \tag{3.132}$$

One can apply this procedure to higher values of N, but we find the approach of the next subsection to be more efficient.

3.6.3 Resumming the large N expansion

Here we take the result of section 3.5 for the exact large N expansions (3.95) and (3.104) of single and two node quivers and resum them for finite values of N. Inspired by the techniques of [92], we find systematic approaches to computing these infinite

[9] This equality can be easily verified by studying the q expansions.

sums.

$\mathcal{N} = 4$ SYM

(3.95) can be easily resummed by applying the expansions (3.110) and (3.115). For even rank $N = 2r$ we thus find

$$\begin{aligned}\mathcal{I}^{\mathcal{N}=4}(2r) &= \frac{q^{\frac{1}{4}}}{\eta^3(\tau)}\sum_{s=-r+1}^{\infty}(-1)^s(2r+2s)\frac{(s+2r-1)!}{s!(2r)!}q^{2rs+s^2}\\ &= \frac{2q^{-r^2+\frac{1}{4}}}{\eta^3(\tau)(2r)!}\sum_{s=-r+1}^{\infty}(-1)^s q^{(s+r)^2}\sum_{j=0}^{r-1}\alpha_j(r)(s+r)^{2r-2j}\\ &= \frac{2q^{-r^2+\frac{1}{4}}(-1)^r}{\eta^3(\tau)(2r)!}\sum_{j=0}^{r-1}\alpha_j(r)\sum_{s=1}^{\infty}(-1)^s q^{s^2}s^{2r-2j}\\ &= \frac{2q^{-r^2+\frac{1}{4}}}{\eta^3(\tau)2^{2r}(2r)!}\sum_{j=0}^{r-1}\alpha_j(r)(-4)^j\vartheta_4^{(2r-2j)}\,,\end{aligned} \tag{3.133}$$

where $\vartheta_i^{(n)} = \partial_z^n\vartheta_i(z,q)|_{z=0}$. Note that in the first line we extended the sum over s to include some negative values compared with (3.95), for which the summand clearly vanishes. Similarly for odd $N = 2r+1$ we find

$$\mathcal{I}^{\mathcal{N}=4}(2r+1) = \frac{q^{-r(r+1)}}{\eta^3(\tau)2^{(2r+1)}(2r+1)!}\sum_{j=0}^{N}\tilde{\alpha}_j(r)(-1)^j\vartheta_1^{(2r-2j+1)}\,. \tag{3.134}$$

These expressions have the advantage over (3.121) that they immediately give expressions for any values of N, but the disadvantage that they involve higher and higher derivatives of theta functions as N becomes large. To recover equivalent expressions involving only elliptic integrals as in (3.121) one can apply the method of footnote 8.

$L = 2$

To resum (3.104) requires a little more effort but we again find a systematic approach. Once more the algorithm has small differences for even and odd ranks, so we first consider (3.104) (without flavour fugacities, i.e. $u = 0$) for even $N = 2r$

$$\mathcal{I}^{L=2}(2r) = \frac{q^{\frac{1}{3}}}{\eta^4(\tau)}\sum_{k=-r+1}^{\infty}\sum_{l=-r+1}^{\infty}(2r+k+l)\frac{(2r+k-1)!(2r+l-1)!}{(2r)!(2r-1)!k!l!}q^{2r(k+l)+2kl}\,. \tag{3.135}$$

Applying the expansion (3.110) and writing the sums over k and l in terms of indices $j = k + r$, $n = l + r$ yields

$$\mathcal{I}^{L=2}(2r) = \frac{q^{\frac{1}{3}}q^{-2r^2}}{\eta^4(\tau)(2r-1)!(2r)!}\sum_{m=0}^{r-1}\sum_{m'=0}^{r-1}\alpha_m(r)\alpha_{m'}(r) \\ \times\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}(j+n)j^{2r-2m-1}n^{2r-2m'-1}q^{2jn}\,. \tag{3.136}$$

We are now faced by (finitely many) double infinite sums of the form

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}j^a n^{a+2s+1}q^{2jn} = \frac{\partial_\tau^a}{(2\pi i)^a}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}n^{2s+1}q^{2jn} = \frac{\partial_\tau^a}{(2\pi i)^a}A_{2s+1}\,, \qquad a, s\in\mathbb{N}\,, \tag{3.137}$$

and likewise with $j \leftrightarrow n$. The quantities $A_{2s+1} = \sum_{n=1}^{\infty} n^{2s+1}\frac{q^{2n}}{1-q^{2n}}$ also appeared in the expansion of Jacobi elliptic functions in [93], and played a central role also in the evaluation of hyperbolic sums in [92]. They are be generated by[10]

$$\sum_{s=0}^{\infty}(-1)^s A_{2s+1}\frac{(2t)^{2s}}{2s!} = \frac{K(K-E)}{4\pi^2} + \frac{1}{8\sin^2 t} - \frac{K^2}{4\pi^2}\,\mathrm{ns}^2\left(\frac{2Kt}{\pi}, k^2\right)\,, \tag{3.138}$$

Arbitrary numbers of τ derivatives of the A_{2s+1} can be easily evaluated by applying the formulas

$$\begin{aligned} \frac{\partial_\tau}{2\pi i}k &= \frac{k(1-k^2)K^2}{\pi^2}\,, \\ \frac{\partial_\tau}{2\pi i}K &= \frac{EK^2-(1-k^2)K^3}{\pi^2}\,, \\ \frac{\partial_\tau}{2\pi i}E &= \frac{(1-k^2)(EK^2-K^3)}{\pi^2}\,. \end{aligned} \tag{3.139}$$

Let us now turn to the case of odd $N = 2r + 1$. Analogously to the even case, the formula (3.115) allows us to write (*c.f.*, (3.136))

$$\mathcal{I}(2r+1) = \frac{q^{\frac{1}{3}}q^{-\frac{(2r+1)^2}{2}}}{\eta^4(\tau)(2r)!(2r+1)!2^{4r}}\sum_{m=0}^{r}\sum_{m'=0}^{r}\tilde{\alpha}_m(r)\tilde{\alpha}_{m'}(r) \\ \times\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}(j+n+1)(2j+1)^{2r-2m}(2n+1)^{2r-2m'}q^{\frac{1}{2}(2j+1)(2n+1)}\,. \tag{3.140}$$

[10] ns and sn used below are standard Jacobi elliptic functions (ns = 1/ sn).

In this case we are faced by double infinite sums

$$\begin{aligned}&\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}(2j+1)^a(2n+1)^{a+2s+1}q^{\frac{1}{2}(2j+1)(2n+1)}\\&\quad=\frac{2^a\partial_\tau^a}{(\pi i)^a}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}(2n+1)^{2s-1}q^{\frac{1}{2}(2j+1)(2n+1)}=\frac{2^a\partial_\tau^a}{(\pi i)^a}H_{2s+1}\,,\end{aligned}\tag{3.141}$$

The quantities $H_{2s+1}=\sum_{n=0}^{\infty}(2n+1)^{2s+1}\frac{q^{n+\frac{1}{2}}}{1-q^{2n+1}}$ also appeared in [92]. They are generated by

$$\sum_{s=0}^{\infty}(-1)^sH_{2s+1}\frac{t^{2s+1}}{(2s+1)!}=\frac{kK}{2\pi}\,\mathrm{sn}\left(\frac{2Kt}{\pi},k^2\right).\tag{3.142}$$

Arbitrary numbers of τ derivatives of the H_{2s+1} can again be straightforwardly evaluated using (3.139).

This algorithm can be easily implemented to sum (3.104) for finite values of N. For $N=1,\dots,4$ this gives

$$\begin{aligned}\mathcal{I}^{L=2}(1)&=\frac{q^{-\frac{1}{6}}k}{\eta^4(\tau)}\frac{K^2}{\pi^2}\,,\\\mathcal{I}^{L=2}(2)&=\frac{q^{-\frac{5}{3}}}{\eta^4(\tau)}\frac{-3E^2K^2+2(2-k^2)EK^3-(1-k^2)K^4}{6\pi^4}\,,\\\mathcal{I}^{L=2}(3)&=\frac{q^{-\frac{25}{6}}k}{\eta^4(\tau)}\Bigg(\frac{6E^2K^4-6(1-k^2)EK^5+(1-k^2)^2K^6}{12\pi^6}-\frac{EK^3+k^2K^4}{24\pi^4}\\&\qquad+\frac{K^2}{192\pi^2}\Bigg)\,,\\\mathcal{I}^{L=2}(4)&=\frac{q^{-\frac{23}{3}}}{\eta^4(\tau)}\Bigg(\frac{-3E^4K^4+4(2-k^2)E^3K^5-6(1-k^2)E^2K^6+(1-k^2)^2K^8}{72\pi^8}\\&+\frac{15E^3K^3-15(2-k^2)E^2K^4+(11-11k^2-4k^4)EK^5+2(1-k^2)(2-k^2)K^6}{1080\pi^6}\\&-\frac{3E^2K^2-2(2-k^2)EK^3+(1-k^2)K^4}{432\pi^4}\Bigg).\end{aligned}\tag{3.143}$$

The algorithm can easily be pushed to higher values of N using Mathematica.

3.6.4 Results from the q-expansion of longer quivers

In all the examples presented above the rescaled index $Z(N)$ (3.27) at finite N can be expressed as a polynomial in K, E and k. This is also true for the trivial case of arbitrary L and $N=1$. This is just the theory of L free hypermultiplets, where the index without flavour fugacities can be rewritten in terms of elliptic integrals as

$$\mathcal{I}^{L>1}(1) = \left(\frac{q^{-\frac{1}{12}}\vartheta_3}{\eta^2(\tau)\vartheta_2}\right)^L Z(1) = \left(\frac{q^{-\frac{1}{12}}k^{\frac{1}{2}}K}{\pi\eta^2(\tau)}\right)^L . \tag{3.144}$$

Inspired by these results, we conjecture that for arbitrary L, N, the rescaled index $Z(N)$ is always given by a polynomial in complete elliptic integrals and the elliptic modulus[11]

$$Z(N) = k^{\frac{L}{2}(1-(-1)^N)} \sum_{j,l,m} a_{j,l,m} k^{2j} \left(\frac{K}{\pi}\right)^l \left(\frac{E}{\pi}\right)^m . \tag{3.145}$$

Studying which terms appear in (3.121), (3.143) and (3.144) we guess that the only nonzero coefficients have

$$\begin{aligned} j \geqslant 0\,, \quad l \geqslant L\,, \quad k \geqslant 0\,, \\ l - m - 2j \geqslant 0\,, \\ LN - l - m \geqslant 0 \quad \text{is even.} \end{aligned} \tag{3.146}$$

These constraints leave us with finitely many $a_{j,l,m}$, which we can fix by comparing the q expansions of each side of (3.145). We first use the relations (3.30), (3.40) and (3.43) to express the left hand side as

$$Z(N) = \sum_{\vec{n}\in\mathbb{Z}^{L-1}} \prod_{p\in\mathbb{Z}} \left(1 + \kappa \prod_{a=1}^{L} \frac{1}{q^{p-n^{(a)}+\frac{1}{2}} + q^{-p+n^{(a)}-\frac{1}{2}}}\right)\Bigg|_{\kappa^N} , \tag{3.147}$$

where $|_{\kappa^N}$ indicates extracting the coefficient of κ^N. Now the q expansion can be easily obtained by truncating the sum over $\vec{n}$ and the product over p at large orders. Solving the resulting linear problems for the $a_{j,l,m}$ and reintroducing the scaling factor in (3.27) we have obtained the results

$$\begin{aligned} \mathcal{I}^{L=3}(2) &= \frac{q^{-\frac{5}{2}}}{\eta^6(\tau)}\left(\frac{-E^3K^3 + 3E^2K^4 - 3(1-k^2)EK^5 + (1-k^2)^2K^6}{2\pi^6} - \frac{k^2K^4}{8\pi^4}\right), \\ \mathcal{I}^{L=3}(3) &= \frac{q^{-\frac{25}{4}}k^{\frac{3}{2}}}{\eta^6(\tau)}\left(\frac{-(1-k^2)^2(1+k^2)K^9}{120\pi^9} - \frac{8EK^4 - (29+21k^2)K^5}{1920\pi^5}\right. \\ &\qquad \left. - \frac{24E^2K^5 - 24(1-k^2)EK^6 + 5(1-k^2)^2K^7}{96\pi^7} + \frac{K^3}{1536\pi^3}\right), \\ \mathcal{I}^{L=4}(2) &= \frac{q^{-\frac{10}{3}}}{\eta^8(\tau)}\left(\frac{-3E^4K^4 + 4(2-k^2)E^3K^5 - 6(1-k^2)E^2K^6 + (1-k^2)^2K^8}{6\pi^8}\right. \\ &\qquad \left. - \frac{2(1-k^2+k^4)EK^5 - (1-k^2)(2-k^2)K^6}{45\pi^6}\right). \end{aligned} \tag{3.148}$$

[11] Note that $q^{\frac{LN^2}{4}} \frac{\vartheta_3^L\left(\frac{\pi\tau N}{2}\right)}{\vartheta_3^L} = k^{\frac{L}{4}(1-(-1)^N)}$.

To fix a unique solution for the first, second and third equalities of (3.148) we required the q expansions of (3.145) up to q^{19}, q^{38} and q^{38} respectively. We have further checked that the solutions reproduce the q expansions of the right hand side of (3.147) up to q^{90}, q^{90} and q^{48} respectively. One could continue to larger values of N and L, but the number of terms required in (3.147) grows very quickly, and so significant computational resources are required.

Appendix A

Combinatorics of the permutations $R\tau^{-1}R\tau$

In this appendix we show how we can simplify the partition function (2.77)

$$Z(N) = \frac{1}{2^{2N}N!^2}\sum_{\tau\in S_{2N}}(-1)^\tau\int d^N\lambda\prod_{k\in\mathcal{K}(\tau)}(-1)^{s(k)+s(\tau(k))}\rho(\lambda_k,\lambda_{R\tau^{-1}R\tau(k)})\,, \qquad \text{(A.1)}$$

by studying more closely the composite permutation $R\tau^{-1}R\tau$ and the set $\mathcal{K}(\tau)$ of N integers in $1,\cdots,2N$ such that $R(\mathcal{K}(\tau)) = \overline{\mathcal{K}(\tau)}$ and $R\tau^{-1}R\tau(\mathcal{K}(\tau)) = \mathcal{K}(\tau)$.

Let us label the N integers in $\mathcal{K}(\tau)$ by $k_1,\cdots,k_N$, such that the action of $R\tau^{-1}R\tau$ on $\mathcal{K}(\tau)$ can be represented in terms of a permutation $\sigma_\tau \in S_N$ as

$$R\tau^{-1}R\tau(k_i) = k_{\sigma_\tau(i)}\,. \qquad \text{(A.2)}$$

We can immediately see that $R\tau^{-1}R\tau$ acts on elements $R(k_i)$ in the compliment of $\mathcal{K}(\tau)$ by the inverse permutation σ_τ^{-1}

$$R\tau^{-1}R\tau(R(k_i)) = R(R\tau^{-1}R\tau)^{-1}(k_i) = R(k_{\sigma_\tau^{-1}(i)})\,. \qquad \text{(A.3)}$$

This property means that $R\tau^{-1}R\tau$ is composed of pairs of cycles that take the form[1]

$$(k_1k_2\cdots k_l)(R(k_l)R(k_{l-1})\cdots R(k_1))\,. \qquad \text{(A.4)}$$

For a given $\sigma \in S_N$ we can easily find $\tau \in S_{2N}$ such that $\sigma = \sigma_\tau$ by for example taking for each l-cycle in σ the $2l$-cycle in τ

$$(k_lR(k_l)k_{l-1}R(k_{l-1})\cdots R(k_1))\,. \qquad \text{(A.5)}$$

This particular choice of τ is useful because $\tau(k_i) = R(k_i)$, and it is made up of only

[1] Indeed there is a freedom in choosing the set $\mathcal{K}(\tau)$ by including the elements of either one of each pair of cycles.

cycles of even length, so we can easily compute

$$(-1)^{\tau} \prod_{k \in \mathcal{K}(\tau)} (-1)^{s(k)+s(\tau(k))} = (-1)^{n_\tau}(-1)^N = (-1)^{n_{\sigma_\tau}}(-1)^N = (-1)^{\sigma_\tau} \,, \qquad \text{(A.6)}$$

where we recall that

$$s(k) = \begin{cases} 0\,, & k = 1, \cdots, N\,, \\ 1\,, & k = N+1, \cdots, 2N\,, \end{cases} \qquad \text{(A.7)}$$

and n_τ counts the number of cycles in τ. The second equality in (A.6) follows because each cycle in τ gives rise to a cycle in σ_τ, and in the third equality we recognised that the expression appearing is nothing but the signature of σ_τ.

Of course, there are many possible choices of τ giving rise to the same $R\tau^{-1}R\tau$ and we should check that all of them reproduce (A.6). From any particular τ we can generate all τ giving rise to the same $R\tau^{-1}R\tau$ by taking

$$\tau \to \pi\tau\,, \qquad \pi^{-1}R\pi = R \qquad \text{(A.8)}$$

To solve this condition, π can be any permutation of the form $\pi = \pi_1\pi_2$, where π_1 is any combination of the two cycles appearing in R (2^N possibilities), and π_2 acts as two copies of some S_N permutation ($N!$ possibilities)

$$\pi_2(i) = \sigma'(i), \qquad \pi_2(N+i) = N + \sigma'(i), \qquad i = 1, \cdots, N, \quad \sigma' \in S_N\,. \qquad \text{(A.9)}$$

It is easy to check that any deformation (A.8) leaves the left-hand side of (A.6) invariant, and so the right-hand side indeed holds for any τ. With this simplification we can rewrite the partition function (A.1) as

$$Z(N) = \frac{1}{2^{2N}N!^2} \sum_{\tau \in S_{2N}} (-1)^{\sigma_\tau} \int d^N\lambda \prod_{i=1}^{N} \rho(\lambda_{k_i}, \lambda_{k_{\sigma_\tau(i)}})\,. \qquad \text{(A.10)}$$

The summand of (A.10) depends only on the conjugacy class of σ_τ, determined by the number of cycles m_l of length l. This means we can convert the sum over S_{2N} permutations to a sum over conjugacy classes of S_N, if we know the combinatorics of the map from τ into the conjugacy class of σ_τ.

We have already counted how many τ give rise to any given $R\tau^{-1}R\tau$ permutation ($2^N N!$), so we just need to compute how many distinct $R\tau^{-1}R\tau$ permutations are associated with a σ_τ of a given conjugacy class. We know that $R\tau^{-1}R\tau$ generates all possible permutations made up of pairs of cycles as in (A.4), and we should count how many of them have m_l pairs of cycles of length l. Suppose we fix how we assign to $k_1, \cdots, k_N$ and $R(k_1), \cdots, R(k_N)$ the integers $1, \cdots, 2N$. With this restriction we are just left with counting the number of ways to distribute the k_i among the cycles, which is the same as counting the number of S_N permutations in a given conjugacy class

$$\frac{N!}{\prod_{l=1}^{N} l^{m_l} m_l!} \,. \tag{A.11}$$

We then have the added possibility of generating more distinct $R\tau^{-1}R\tau$ (without altering the conjugacy class) by exchanging $k_i \leftrightarrow R(k_i)$. Note however that simultaneously exchanging all of the k_i within a given cycle leaves $R\tau^{-1}R\tau$ invariant, so this generates just an additional factor of

$$\frac{2^N}{2^{n_{\sigma_\tau}}} \,, \tag{A.12}$$

Putting all this together and relabelling $\lambda_{k_i} \to \lambda_i$, (A.10) becomes

$$Z(N) = \frac{1}{N!} \sum_{\sigma \in S_N} \frac{(-1)^\sigma}{2^{n_\sigma}} \int d^N\lambda \prod_{i=1}^{N} \rho(\lambda_i, \lambda_{\sigma(i)}) \,, \tag{A.13}$$

where we have absorbed the factor (A.11) to promote the sum over S_N conjugacy classes to a sum over S_N permutations.

Appendix B

Degeneracy of the spectrum

In this appendix we show that for vanishing masses and FI parameters the expressions for the partition functions found in the main text in terms of density operators (2.81) (which is identical to (A.13) above) correspond to the partition function of fermions on a semi-infinite line (2.83). We do this by proving that the spectrum of ρ splits into odd and even states whose spectrum is identical.[1]

The first statement amounts to ρ commuting with the reflection operator $\hat{R}$ which acts on states by $\hat{R}\,|\lambda\rangle = |-\lambda\rangle$. For vanishing masses and FI parameters the density operator ρ in (2.98) is given by a sequence of even or odd functions of p or q, with precisely two odd functions. Since $\hat{R}f(q) = f(-q)\hat{R}$, $\hat{R}f(p) = f(-p)\hat{R}$, for any function f, we see that $\hat{R}$ commutes with ρ.

To show that the spectrum of odd and even states is the same, we prove that $\mathrm{Tr}(\rho^l \hat{R}) = 0$ for all l. Let us focus first on the case $l = 1$. We notice that the density operator takes the specific form

$$\rho = B_D^{(0)}(p,q)\,\tilde{\rho}\,B_D^{(L)}(p,q)\,\tilde{\rho}^\dagger\,, \tag{B.1}$$

with $\tilde{\rho}$ a sequence of even functions of p or q, $\tilde{\rho}^\dagger$ is the Hermitean conjugate and

$$B_D^{(a)}(p,q) = \left(F^{(a)}(q)\frac{\mathrm{sh}\,p}{\mathrm{ch}\,p}F'^{(a)}(q) + F'^{(a)}(q)\frac{\mathrm{sh}\,p}{\mathrm{ch}\,p}F^{(a)}(q)\right), \qquad a = 0, L\,. \tag{B.2}$$

Using the properties[2]

$$\hat{R}^\dagger = \hat{R}\,, \quad f(q)^\dagger = f(q)\,, \quad f(p)^\dagger = f(-p)\,, \tag{B.3}$$

[1] Apart for a single even zero-mode, which is non-normalisable.

[2] Here † acts as transposition, since we consider only real operators.

we can derive the chain of equalities

$$\begin{aligned}\mathrm{Tr}(\rho\hat{R}) &= \mathrm{Tr}\left((\rho\hat{R})^\dagger\right) = \mathrm{Tr}\left(\hat{R}\rho^\dagger\right) = \mathrm{Tr}\left(\hat{R}\,\tilde{\rho}\,B_D^{(L)}(p,q)\,\tilde{\rho}^\dagger\,B_D^{(0)}(p,q)\right) \\ &= -\,\mathrm{Tr}\left(B_D^{(0)}(p,q)\,\tilde{\rho}\,B_D^{(L)}(p,q)\,\tilde{\rho}^\dagger\,\hat{R}\right) = -\,\mathrm{Tr}(\rho\hat{R})\,,\end{aligned} \tag{B.4}$$

where we have used $(B_D^{(a)}(p,q))^\dagger = -B_D^{(a)}(p,q)$, the cyclicity of the trace and commuted $\hat{R}$ and $B_D^{(0)}(p,q)$, producing a minus sign. This yieds $\mathrm{Tr}(\rho\hat{R}) = 0$. The argument generalises easily for $l \geqslant 2$.

To derive (2.81), notice that the effect of the projection $\frac{1\pm\hat{R}}{2}$ in (2.83), (2.84) is to add a factor of 1/2 to every cycle in a given permutation $\sigma \in S_N$,

$$\begin{aligned}\int\prod_{k=1}^{l} d\lambda_{i_k}\,\langle\lambda_{i_l}|\,\rho\left(\frac{1\pm\hat{R}}{2}\right)|\lambda_{i_1}\rangle\prod_{k=1}^{l-1}\langle\lambda_{i_k}|\,\rho\left(\frac{1\pm\hat{R}}{2}\right)|\lambda_{i_{k+1}}\rangle &= \mathrm{Tr}\left(\left(\rho\,\frac{1\pm\hat{R}}{2}\right)^l\right) \\ &= \mathrm{Tr}\left(\rho^l\,\frac{1\pm\hat{R}}{2}\right) = \frac{1}{2}\,\mathrm{Tr}\left(\rho^l\right).\end{aligned} \tag{B.5}$$

The same results hold for density operators of linear quivers (2.99) at vanishing masses and FI parameters. The arguments are the same except that one must consider the momentum space basis $|p\rangle$, with for instance $\mathrm{Tr}\,\hat{A} = \int dp\,\langle p|\,\hat{A}\,|p\rangle$.

Appendix C

Relation with previous Fermi gas formulation of $Sp(2N)$ quivers

In this appendix we compare our results to those in [36], which also developed a Fermi gas approach to Sp quivers. As we show below, the two formulations are equivalent, but in the final expression for the coefficient B in the Airy function we disagree with their results and explain why.

The overlap between theories discussed in [36] and our work is the single node Sp quiver with an antisymmetric hypermultiplet and n fundamental hypermultiplets. The partition function is given by the matrix model

$$Z(N) = \frac{1}{4^N N!} \int d^N\lambda \prod_{i=1}^{N} \frac{\operatorname{sh}^2 2\lambda_i \operatorname{ch} 2\lambda_i}{\operatorname{ch}^{2n} \lambda_i} \frac{\prod_{i<j} \operatorname{sh}^2(\lambda_i - \lambda_j) \operatorname{sh}^2(\lambda_i + \lambda_j)}{\prod_{i,j} \operatorname{ch}(\lambda_i - \lambda_j) \operatorname{ch}(\lambda_i + \lambda_j)} . \tag{C.1}$$

In [36] the matrix model (C.1) was manipulated with a modified Cauchy identity

$$\frac{\prod_{i<j}(x_i - x_j)(y_i - y_j)(x_i x_j - 1)(y_i y_j - 1)}{\prod_{i,j}(x_i + y_j)(x_i y_j + 1)} = \sum_{\sigma} (-1)^{\sigma} \prod_i \frac{1}{(x_i + y_{\sigma(i)})(x_i y_{\sigma(i)} + 1)} . \tag{C.2}$$

This identity, with the usual the usual replacements $x_i \to e^{\lambda_i}$, $y_i \to e^{\lambda_i}$, gives

$$Z(N) = \frac{1}{4^N N!} \sum_{\sigma \in S_N} (-1)^{\sigma} \int d^N\lambda \prod_{i=1}^{N} \frac{\prod_i \operatorname{sh}^2 2\lambda_i \operatorname{ch} 2\lambda_i}{\prod_i \operatorname{ch}^{2n} \lambda_i} \frac{1}{\operatorname{ch}(\lambda_i - \lambda_{\sigma(i)}) \operatorname{ch}(\lambda_i + \lambda_{\sigma(i)})} \tag{C.3}$$

Using the relations[1]

$$\begin{aligned} \frac{\operatorname{ch} \pi\lambda_1 \operatorname{ch} \lambda_2}{\operatorname{ch}(\lambda_1 - \lambda_2) \operatorname{ch}(\lambda_1 + \lambda_2)} &= \langle \lambda_1 | \frac{1 + \hat{R}}{2 \operatorname{ch} p} | \lambda_2 \rangle , \\ \frac{\operatorname{sh} \lambda_1 \operatorname{sh} \lambda_2}{\operatorname{ch}(\lambda_1 - \lambda_2) \operatorname{ch}(\lambda_1 + \lambda_2)} &= \langle \lambda_1 | \frac{1 - \hat{R}}{2 \operatorname{ch} p} | \lambda_2 \rangle , \end{aligned} \tag{C.4}$$

[1] Note that $\hat{R}$ commutes with $\frac{1}{\operatorname{ch} p}$, and so we can write $\frac{1+\hat{R}}{2 \operatorname{ch} p}$ ($= \frac{1}{\operatorname{ch} p} \frac{1+\hat{R}}{2}$) which would otherwise be ill defined

from which one immediately has the corollary

$$\frac{1-\hat{R}}{2\operatorname{ch}p} = \frac{\operatorname{sh}q}{\operatorname{ch}q}\frac{1+\hat{R}}{2\operatorname{ch}p}\frac{\operatorname{sh}q}{\operatorname{ch}q}\,, \tag{C.5}$$

they rewrite the partition function in two equivalent forms

$$Z(N) = \frac{1}{N!}\sum_{\sigma\in S_N}(-1)^\sigma \int \prod_{i=1}^{N} d\lambda_i \prod_{i=1}^{N} \langle\lambda_i|\,\rho_\pm\left(\frac{1\pm\hat{R}}{2}\right)|\lambda_{\sigma(i)}\rangle\,, \tag{C.6}$$

where

$$\begin{aligned}\rho_+ &= \frac{\operatorname{sh}^2 q\operatorname{ch}2q}{\operatorname{ch}^{2n}q}\frac{1}{\operatorname{ch}p}\,,\\ \rho_- &= \frac{\operatorname{ch}2q}{\operatorname{ch}^{2n-2}q}\frac{1}{\operatorname{ch}p}\,.\end{aligned} \tag{C.7}$$

Indeed, this looks very similar to our rewriting of the matrix model as (2.83) or (2.84)

$$Z(N) = \frac{1}{N!}\sum_{\sigma\in S_N}(-1)^\sigma \int \prod_{i=1}^{N} d\lambda_i \prod_{i=1}^{N} \langle\lambda_i|\,\rho\left(\frac{1\pm\hat{R}}{2}\right)|\lambda_{\sigma(i)}\rangle\,, \tag{C.8}$$

but with a different density operator (2.103)

$$\rho = \frac{1}{2}\frac{\operatorname{sh}2q}{\operatorname{ch}^{2n}q}\left(\operatorname{sh}q\frac{1}{\operatorname{ch}p}\operatorname{ch}q + \operatorname{ch}q\frac{1}{\operatorname{ch}p}\operatorname{sh}q\right). \tag{C.9}$$

All those expressions are in fact equivalent, since we can show that the spectrum of even states of ρ agrees with that of ρ_+ and the spectrum of odd states to that of ρ_-. Indeed the projected operators are similar to each other by the simple manipulations

$$\begin{aligned}\rho\,\frac{1-\hat{R}}{2} &= \frac{1}{2}\frac{\operatorname{sh}2q}{\operatorname{ch}^{2n}q}\left(\operatorname{sh}q\frac{1-\hat{R}}{2\operatorname{ch}p}\operatorname{ch}q + \operatorname{ch}q\frac{1+\hat{R}}{2\operatorname{ch}p}\operatorname{sh}q\right)\\ &= \frac{1}{2}\frac{\operatorname{sh}2q}{\operatorname{ch}^{2n}q}\left(\operatorname{sh}q\frac{1-\hat{R}}{2\operatorname{ch}p}\operatorname{ch}q + \frac{\operatorname{ch}^2 q}{\operatorname{sh}q}\frac{1-\hat{R}}{2\operatorname{ch}p}\operatorname{ch}q\right)\\ &= \frac{\operatorname{ch}2q\operatorname{ch}q}{\operatorname{ch}^{2n}q}\frac{1-\hat{R}}{2\operatorname{ch}p}\operatorname{ch}q = \frac{1}{\operatorname{ch}q}\left(\rho_-\frac{1-\hat{R}}{2}\right)\operatorname{ch}q\,,\end{aligned} \tag{C.10}$$

where in the first line we used that $\hat{R}$ commutes with even and anticommutes with odd functions of q, and in the second line we used (C.5). Similar manipulations lead to the relations

$$\frac{1}{\operatorname{ch}q}\left(\rho_-\frac{1-\hat{R}}{2}\right)\operatorname{ch}q = \frac{1}{\operatorname{sh}q}\left(\rho_+\frac{1+\hat{R}}{2}\right)\operatorname{sh}q = \frac{\operatorname{ch}q}{\operatorname{ch}2q\operatorname{sh}q}\left(\rho\,\frac{1+\hat{R}}{2}\right)\frac{\operatorname{ch}2q\operatorname{sh}q}{\operatorname{ch}q}\,. \tag{C.11}$$

We can also derive the Airy function expression for these theories based on our analysis in sections 2.5 and 2.6. The mirrors of this class of theories are $\hat{D}$ quivers with $(n-3)$ $U(2N)$ nodes and and a single fundamental hypermultiplet on one $U(N)$ node. The asymptotic expansion of the grand potential (2.186) for this theory ($L = n-2$, $\nu = \Delta = \frac{1}{2}$, $\Sigma_1 = \Sigma_2 = 0$) is given by[2]

$$C = \frac{1}{2\pi^2(n-2)}\,, \qquad B = \frac{1}{8}\Big(-n-1+\frac{1}{n-2}\Big). \tag{C.12}$$

The coefficient C is the same as found in [36], but B is not. The reason for the discrepancy is that in our formulation, the operator ρ has degenerate odd/even spectrum, as shown in appendix B. While the odd and even spectra agree with ρ_+ and ρ_- of [36], those latter operators do not have degenerate spectra. The saddle point calculation of [36] based on the Fermi surface of ρ_- considered the full spectrum of this operator and divided the end result by 2. Since the operator does not have a degenerate spectrum, this is merely an approximation that is good enough to evaluate the leading order term C, but fails for the subleading coefficient B.

[2]We thank Silviu Pufu for correcting a mistake in this formula.

Appendix D

Truncation of the ϵ expansion

Here we show that the ϵ expansions of C and B coefficients in (2.154) truncates at order ϵ^0 and ϵ^2 respectively, adapting a similar proof from [14] for $\hat{A}$ quiver theories.

We recall that corrections to C, B and A coefficients at order ϵ^n can be computed from a single residue involving $Z_{l(n)}$

$$J(\mu) = \sum_{n \geqslant 0} \epsilon^n J_{(n)}(\mu), \qquad J_{(n)}(\mu) = -\frac{1}{2} \operatorname*{Res}_{l=0} \Gamma(l)\Gamma(-l) Z_{l(n)} e^{l\mu} + \mathcal{O}(e^{-\alpha\mu}), \qquad \alpha > 0 \,. \tag{D.1}$$

From this expression it is clear that if $Z_{l(n)}$ vanishes at $l = 0$, then $J_{(n)}(\mu)$ can only correct the A coefficient. We should prove then that $Z_{l(n)}$ has at least a simple zero at $l = 0$ for all $n > 2$.

It is useful to consider the Wigner-Kirkwood expansion of Z_l [14]. The idea is to express Z_l in terms of the Fermi gas Hamiltonian, $H_W = -\log_\star \rho_W$

$$Z_l = \sum_{r \geqslant 0} \frac{(-l)^r}{r!} \int dpdq \, e^{-lH_W} \mathcal{G}_r, \qquad \mathcal{G}_r = \left((\hat{H} - H_W)^r \right)_W . \tag{D.2}$$

We recall that the $\hat{D}$ density operator is given by (2.170)

$$\begin{aligned} \rho_W = \sqrt[\star]{e^{S(p)+2U_\lambda(q)}} \star e^{T(p)} \star \left(\prod_{k=1}^{\lambda-1}{}_\star \, e^{U_{\lambda-k}(q)} \star e^{T(p)} \right) \star e^{S(p)+2U_0(q)} \\ \star \left(\prod_{k=1}^{\lambda-1}{}_\star \, e^{T(p)} \star e^{U_k(q)} \right) \star e^{T(p)} \star \sqrt[\star]{e^{S(p)+2U_\lambda(q)}} \,. \end{aligned} \tag{D.3}$$

Since (D.3) has an expansion in purely even powers of ϵ, H_W likewise takes the form

$$H_W = H_{(0)} + \epsilon^2 H_{(2)} + \epsilon^4 H_{(4)} + \cdots . \tag{D.4}$$

This gives

$$Z_l = \sum_{r\geqslant 0} \frac{(-l)^r}{r!} \int dpdq\, e^{-lH_{(0)}} \left(1 - l\sum_{n\geqslant 1} \epsilon^{2n} H_{(2n)} + \frac{l^2}{2}\left(\sum_{n\geqslant 1} \epsilon^{2n} H_{(2n)}\right)^2 + \mathcal{O}(l^3)\right) \mathcal{G}_r\,. \tag{D.5}$$

We know that all of the integrals that appear in this expansion are of the form (2.180)

$$\int dx \frac{\mathrm{th}^a\, x}{\mathrm{ch}^b\, x} = \frac{(1+e^{i\pi a})\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b}{2}\right)}{2^{b+1}\pi\Gamma\left(\frac{a+b+1}{2}\right)}\,. \tag{D.6}$$

The coefficients a and b are given by linear functions of l and so the integrals over p and q in (D.5) can each produce at most a simple pole at $l=0$, from the $\Gamma\left(\frac{b}{2}\right)$. Therefore, we need only concern ourselves with terms in (D.5) with prefactors of at most order l^2. Discarding also (most of) the terms of order ϵ^2 or less, we are left with[1]

$$\begin{aligned}\int dpdq\, e^{-lH_{(0)}} \left(-l\sum_{n\geqslant 2} \epsilon^{2n} H_{(2n)} + \frac{l^2}{2}\left(\sum_{n\geqslant 1} \epsilon^{2n} H_{(2n)}\right)^2\right) \\ + \frac{l^2}{2}\int dpdq e^{-lH_{(0)}}(H_W \star H_W - H_W^2)\,,\end{aligned} \tag{D.7}$$

For simplicity let us now restrict the density operator (D.3) to cases with $U_0(q) = U_\lambda(q) = 0$.[2] With this restriction, the exponentials in (D.3) can be freely exchanged with star exponentials, and we can straightforwardly evaluate the star logarithm by the star product version of the Baker-Campbell-Hausdorff expansion. The leading term gives

$$e^{-lH_{(0)}} = \left(\frac{\mathrm{th}^2\, p}{\mathrm{ch}^{2L}\, p\, \mathrm{ch}^{2\nu}\, q}\right)^l\,. \tag{D.8}$$

All ϵ corrections to H_W are then given by nested star commutators involving $T(p)$, $S(p)$ and $U_k(q)$. An example of such a term contributing to $H_{(4)}$ would be

$$[T(p),[T(p),[T(p),[T(p),U_k(q)]]]]_\star = \frac{\epsilon^4}{16\pi^4}\dot{T}^4 U_k^{(4)} + \mathcal{O}(\epsilon^6)\,, \tag{D.9}$$

where we have used

$$[f,g]_\star = f\star g - g\star f = i\frac{\epsilon}{2\pi}\{f,g\} + \text{higher derivative terms}\,. \tag{D.10}$$

As illustrated in (D.9), in order to have a term at order ϵ^n with only single derivatives

[1] The final term $(H_W \star H_W - H_W^2)$ has still some ϵ^2 piece, but as we shall see this also has at least a simple zero at $l=0$

[2] This corresponds to restricting the $\hat{D}$ quiver theories shown in figure 2.2 to a subclass with $n^{(0)} = n'^{(0)}$ and $n^{(L)} = n'^{(L)}$. We expect that theories outside this subclass also have B and C coefficients truncating at ϵ^2. We have verified that this holds true up to ϵ^4.

acting on functions of p (or q), such a term has a single function of q (or p) with an n^{th} derivative.

This is important, because every term in the epsilon expansion has therefore at least one multiple derivative of S, T or U_k. From (2.168) it follows that these derivatives take the form

$$U_k^{(n)}(q) = \frac{1}{\operatorname{ch}^2 q} \sum_{a\in\mathbb{Z}} \sum_{b\geqslant 0} C_{ab} \frac{\operatorname{th}^a q}{\operatorname{ch}^b q}\,, \qquad n \geqslant 2\,, \tag{D.11}$$

and similarly for S, T with $q \to p$.

We now should combine these derivative terms with (D.8) in (D.7), and integrate with (D.6). It is clear that since the derivative terms contribute always a $\frac{1}{\operatorname{ch}^2 q}$ or $\frac{1}{\operatorname{ch}^2 p}$, the resulting Gamma functions can contribute at most a simple pole at $l = 0$. This guarantees that the terms in (D.7) with an l^2 outside have at least a simple zero, leaving us with

$$-\,l \int dpdq\, e^{-lH_{(0)}} \sum_{n\geqslant 2} \epsilon^{2n} H_{(2n)}\,. \tag{D.12}$$

By the same reasoning, terms with multiple derivatives on both a function of p and a function of q have an overall $\frac{1}{\operatorname{ch}^2 q}\frac{1}{\operatorname{ch}^2 p}$ which kills both of the poles one could get from integrating. The remaining terms in (D.12) which don't obviously have a simple zero are those where all derivatives on functions of p (or q) are first order, like (D.9). But such terms can be integrated by parts; for instance (D.9) would give

$$-\,l \int dpdq\, e^{-lH_{(0)}} \frac{\epsilon^4}{16\pi^4} \dot{T}^4 U_k^{(4)} = l^2 \int dpdq\, e^{-lH_{(0)}} \frac{\epsilon^4}{16\pi^4} \dot{T}^4 U_k^{(3)} H'_{(0)}\,. \tag{D.13}$$

Integrating by parts pulls down an additional factor of l, which guarantees that there is an overall simple zero at $l = 0$, since the integral on the right hand side still produces just a simple pole. Since we have shown that at order ϵ^4 and higher Z_l has at least a simple zero at $l = 0$, this concludes the proof that C and B do not get contributions beyond order ϵ^2.

Appendix E

Definitions and useful identities for elliptic functions

In our study of Schur indices, we chose to use Jacobi theta functions and the Dedekind eta function rather than q-theta functions and q-Pochhammer symbols. These are related by

$$\theta(e^{2iz}, q^2) = \frac{-ie^{iz}\,\vartheta_1(z,q)}{q^{1/6}\eta(\tau)}\,, \qquad \left(q^2;q^2\right)_\infty = q^{-1/12}\eta(\tau) = \prod_{k=1}^{\infty}\left(1-q^{2k}\right)\,. \tag{E.1}$$

where the (quasi)period τ is related to the nome q by $q = e^{i\pi\tau}$. The Jacobi theta functions $\vartheta_3(z,q)$ and $\vartheta_2(z,q)$ are given by the series and product representations

$$\begin{aligned}
\vartheta_3(z,q) &= \sum_{n=-\infty}^{\infty} q^{n^2} e^{2inz} = \prod_{k=1}^{\infty}\left(1-q^{2k}\right)\left(1+2q^{2k-1}\cos(2z)+q^{4k-2}\right)\,,\\
\vartheta_2(z,q) &= q^{\frac{1}{4}}e^{-iz}\vartheta_3(z-\tfrac{1}{2}\pi\tau,q) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}e^{i(2n+1)z}\\
&= 2q^{\frac{1}{4}}\cos(z)\prod_{k=1}^{\infty}\left(1-q^{2k}\right)\left(1+2q^{2k}\cos(2z)+q^{4k}\right)\,.
\end{aligned} \tag{E.2}$$

The remaining two theta functions are given by

$$\begin{aligned}
\vartheta_1(z,q) &= iq^{\frac{1}{4}}e^{-iz}\vartheta_3(z-\tfrac{1}{2}\pi\tau-\tfrac{1}{2}\pi,q)\,,\\
\vartheta_4(z,q) &= \vartheta_3(z-\tfrac{1}{2}\pi,q)\,.
\end{aligned} \tag{E.3}$$

ϑ_3 satisfies the quasi-periodic properties for any integers n, m

$$\vartheta_3(z+n\pi+m\pi\tau,q) = q^{-m^2}e^{-2izm}\vartheta_3(z,q)\,. \tag{E.4}$$

We also give here formulae to evaluate integrals of derivatives of theta functions

$$\begin{aligned}\frac{1}{2\pi i}\int_{-i\pi}^{i\pi} d\mu e^{-m\mu}\partial_\mu^l \vartheta_3(i\mu,q) &= m^l q^{\frac{m^2}{4}}\frac{1}{2}(1+(-1)^m)\,,\\ \frac{1}{2\pi i}\int_{-i\pi}^{i\pi} d\mu e^{-m\mu}\partial_\mu^l \vartheta_2(i\mu,q) &= m^l q^{\frac{m^2}{4}}\frac{1}{2}(1-(-1)^m)\,.\end{aligned} \tag{E.5}$$

Jacobi's imaginary transformation with $\tau = -1/\tau'$, and $q' \equiv e^{i\pi\tau'}$ are

$$\begin{aligned}\vartheta_1(z,q) &= (-i)(-i\tau)^{-\frac{1}{2}}e^{i\tau' z^2/\pi}\vartheta_1(\tau' z,q')\,,\\ \vartheta_2(z,q) &= (-i\tau)^{-\frac{1}{2}}e^{i\tau' z^2/\pi}\vartheta_4(\tau' z,q')\,,\\ \vartheta_3(z,q) &= (-i\tau)^{-\frac{1}{2}}e^{i\tau' z^2/\pi}\vartheta_3(\tau' z,q')\,,\\ \vartheta_4(z,q) &= (-i\tau)^{-\frac{1}{2}}e^{i\tau' z^2/\pi}\vartheta_2(\tau' z,q')\,.\end{aligned} \tag{E.6}$$

We also use in the main text the formula

$$\vartheta_2\vartheta_3\vartheta_4 = 2\eta(\tau)^3\,, \tag{E.7}$$

as well as (see 20.7(iv) of [94])

$$\eta^2(\tau/2) = \vartheta_4\eta(\tau)\,. \tag{E.8}$$

We also require Watson's identity (see 20.7(v) of [94])

$$\vartheta_3(z,q)\vartheta_3(\omega,q) = \vartheta_3\big(z+\omega,q^2\big)\vartheta_3\big(z-\omega,q^2\big) + \vartheta_2\big(z+\omega,q^2\big)\vartheta_2\big(z-\omega,q^2\big)\,. \tag{E.9}$$

An infinite sum in terms of Jacobi theta functions

We make use in Section 3.4.2 of the formula

$$\sum_{n=1}^{\infty}(-1)^n\frac{\cos 4\alpha n}{n\sinh(-i\pi\tau n)} = -\frac{i\pi\tau}{12} - \frac{1}{6}\log\frac{4}{kk'} - \log\frac{\vartheta_3(2\alpha,q)}{\vartheta_3(0,q)}\,, \tag{E.10}$$

which is a combination of (see 16.30.3 of [95])

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n\sinh(-i\pi\tau n)}(1-\cos 4\alpha n) = \log\frac{\vartheta_3(2\alpha,q)}{\vartheta_3(0,q)}\,, \tag{E.11}$$

and (see T1.3 of [96])

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n\sinh(-i\pi n\tau)} = -\frac{i\pi\tau}{12} - \frac{1}{6}\log\frac{4}{kk'}\,, \tag{E.12}$$

where the elliptic modulus and complementary elliptic modulus are respectively defined as

$$k = \frac{\vartheta_2^2}{\vartheta_3^2}\,, \quad k' = \frac{\vartheta_4^2}{\vartheta_3^2}\,. \tag{E.13}$$

A multiple angle formula for Jacobi theta functions

We prove here a formula for the product of theta functions shifted by roots of unity used in Section 3.4.1[1]

$$\begin{aligned}
&\prod_{j=1}^{L} \vartheta_3\big(z + \tfrac{L-2j+1}{2L}\pi, q\big) \\
&\qquad = \prod_{n=1}^{\infty}\prod_{j=1}^{L}(1-q^{2n})(1+2q^{2n-1}\cos\big(2z+\tfrac{\pi}{L}(L+2j-1)\big)+q^{4n-2}) \\
&\qquad = \prod_{n=1}^{\infty}\prod_{j=1}^{L}(1-q^{2n})\big(1+e^{i\pi\frac{L-2j+1}{L}+2iz}q^{2n-1}\big)\big(1+e^{-i\pi\frac{L-2j+1}{L}-2iz}q^{2n-1}\big) \\
&\qquad = \prod_{n=1}^{\infty}(1-q^{2n})^L\big(1+e^{2iLz}q^{L(2n-1)}\big)\big(1+e^{-2iLz}q^{L(2n-1)}\big) \\
&\qquad = \vartheta_3(Lz, q^L)\frac{\eta^L(\tau)}{\eta(L\tau)}\,.
\end{aligned} \tag{E.14}$$

[1]This formula can also be found (without proof) in [97].

Appendix F

A determinant identity for Jacobi theta functions

A crucial identity for our analysis of 4d indices is the generalisation of the Cauchy determinant identity to theta functions. For arbitrary x_i, y_j, t with $i, j = 1, \cdots, n$ we have the identity for q-theta functions [98, 99]

$$\det_{ij}\left(\frac{\theta(tx_jy_i)}{\theta(t)\theta(x_jy_i)}\right) = \frac{\theta(tx_1x_2\cdots x_ny_1y_2\cdots y_n)}{\theta(t)}\frac{\prod_{i<j}x_jy_j\theta(x_i/x_j)\theta(y_i/y_j)}{\prod_{i,j}\theta(x_jy_i)}\,, \tag{F.1}$$

where we have used the notation $\theta(z) = \theta(z, q^2)$.

One can recover the usual Cauchy identity by taking the limit $q \to 0$, where $\theta(z) \to 1 - z$. Taking also the limit $t \to \infty$ gives

$$\det_{ij}\left(\frac{1}{1-x_iy_j}\right) = \frac{\prod_{i<j}(x_i - x_j)(y_j - y_i)}{\prod_{i,j}(1 - x_iy_j)}\,, \tag{F.2}$$

and the usual form of the Cauchy identity is recovered by taking $x_i \to \frac{1}{x_i}$.

In the study of indices we encounter a determinant closely related to (F.1). Making the replacement $x_i \to e^{2i\alpha_i}$, $y_i \to qe^{-2i\alpha'_i}$ as well as $t \to -q^{2T}$, and rewriting the expression in terms of Jacobi theta functions yields

$$\begin{aligned} &\frac{\prod_{i<j}\vartheta_1(\alpha_i-\alpha_j)\vartheta_1(\alpha'_i-\alpha'_j)}{\prod_{i,j=1}^N\vartheta_4(\alpha_i-\alpha'_j)} \\ &= \det_{ij}\left(\frac{\vartheta_3(\alpha_i-\alpha'_j+\pi\tau T)}{\vartheta_4(\alpha_i-\alpha'_j)}\right)\frac{q^{-\frac{N^2}{4}-NT}}{\vartheta_2\left(\sum_{i=1}^N(\alpha_i-\alpha'_i)+\pi\tau(T+\frac{N}{2})\right)}\frac{e^{-iN\sum_{i=1}^N(\alpha_i-\alpha'_i)}}{\vartheta_2(\pi\tau T)^{N-1}}\,. \end{aligned} \tag{F.3}$$

By choosing $T = -\frac{1}{2}$ we obtain

$$\det_{ij}\left(\frac{\vartheta_2(\alpha_i-\alpha'_j)}{\vartheta_4(\alpha_i-\alpha'_j)}\right)\frac{q^{-\frac{N^2}{4}}}{\vartheta_3\left(\sum_{i=1}^N(\alpha_i-\alpha'_i)+N\frac{\pi\tau}{2}\right)}\frac{e^{-iN\sum_{i=1}^N(\alpha_i-\alpha'_i)}}{\vartheta_3^{N-1}(0)}\,. \tag{F.4}$$

The ratio of Jacobi theta functions appearing in the determinant is in fact closely related to the Jacobi elliptic function cn

$$\frac{\vartheta_2(z)}{\vartheta_4(z)} = \frac{\vartheta_2}{\vartheta_4}\,\mathrm{cn}(z\vartheta_3^2)\,, \tag{F.5}$$

where $\mathrm{cn}(z) \equiv \mathrm{cn}(z, k^2)$ and the elliptic modulus k is defined in (E.13).

Appendix G

An expansion formula

Here we present a proof for

$$\left(\tfrac{\kappa+q+q^{-1}+\sqrt{(\kappa+q+q^{-1})^2-4}}{2}\right)^x = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{x(x-k-1)!(x-l-1)!}{k!l!(x-k-l)!(x-k-l-1)!}\kappa^{x-k-l}q^{k-l}\,. \tag{G.1}$$

Our starting point is the expansion

$$\left(\frac{y+\sqrt{y^2-4}}{2}\right)^x = \sum_{s=0}^{\infty}\frac{(-1)^s x(x-s-1)!}{s!(x-2s)!}y^{x-2s}\,. \tag{G.2}$$

Replacing $y^{x-2s} = (\kappa+q+q^{-1})^{x-2s}$ by its multinomial expansion gives

$$\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^s x(x-s-1)!}{s!m!j!(x-2s-m-j)!}\kappa^{x-2s-m-j}q^{j-m}\,. \tag{G.3}$$

Rewriting the sum in terms of indices $l = m+s$ and $k = j+s$ gives

$$\sum_{s=0}^{\infty}\sum_{k=s}^{\infty}\sum_{l=s}^{\infty}\frac{(-1)^s x(x-s-1)!}{s!(k-s)!(l-s)!(x-k-l)!}\kappa^{x-k-l}q^{k-l}\,. \tag{G.4}$$

Interchanging the order of summation we finally obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{s=0}^{\min(l,k)}&\frac{(-1)^s x(x-s-1)!}{s!(k-s)!(l-s)!(x-k-l)!}\kappa^{x-k-l}q^{k-l}\\ &=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{x(x-k-1)!(x-l-1)!}{k!l!(x-k-l)!(x-k-l-1)!}\kappa^{x-k-l}q^{k-l}\,.\end{aligned} \tag{G.5}$$

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